OPTIMISTIC AND PESSIMISTIC APPROACHES FOR COOPERATIVE GAMES

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ABSTRACT. Cooperative game theory aims to study how to divide a joint value created by a set of players. These games are often studied through the characteristic function form with transferable utility, which represents the value obtainable by each coalition. In the presence of externalities, there are many ways to define this value. Various models that account for different levels of player cooperation and the influence of external players on coalition value have been studied. Although there are different approaches, typically, the optimistic and pessimistic approaches provide sufficient insights into strategic interactions. This paper clarifies the interpretation of these approaches by providing a unified framework. We show that making sure that no coalition receives more than their (optimistic) upper bounds is always at least as difficult as guaranteeing their (pessimistic) lower bounds. We also show that if externalities are negative, providing these guarantees is always feasible. Then, we explore applications and show how our findings can be applied to derive results from the existing literature.

Keywords: Cooperative games · optimization problems · cost sharing · core · anti-core · externalities

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1. Introduction

Cooperative game theory primarily focuses on groups of players collaborating and pooling their profits/costs. A central question in this field is how the profits (or costs) of a joint effort can be divided among the members of these coalitions. A cooperative game is a characteristic function that specifies the value obtainable by each coalition of players. We focus on games with transferable utility (TU), which assume that each coalition can distribute its value in any way among its members. [1]

It is, however, not always straightforward to determine the value that should be credited to a coalition, as the behavior of other agents can influence this value. Various models have emerged to address the spectrum of player cooperation levels. These models include multi-choice games (Hsiao and Raghavan, 1993), where players exhibit differing degrees of partial cooperation. Additionally, fuzzy cooperative games (Aubin, 1981) consider a range of participation levels for players, spanning from non-cooperation to full cooperation. Furthermore, partition function form games (Kóczy, 2018) account for all the ways in which the behavior of players outside the coalition influences the value attainable by a coalition.

While these contributions allow to account for the behavior of external players, it is often enough to limit the analysis to two opposing and extreme perspectives to sufficiently capture the dynamics of strategic interactions: the optimistic and pessimistic approaches. The details of these approaches depend on the specific characteristics of the game under consideration.

Along these lines, Curiel and Tijs (1991) introduced two operators, minimarg and maximarg, which determine each coalition's marginal contribution based on the worst and best possible order of agents, respectively. The minimarg assigns the smallest marginal contribution, while the maximarg assigns the largest, embodying pessimistic and optimistic viewpoints, respectively. Iteratively applying these operators to a game leads to a convex and concave game in the limit, with these games being dual to each other. Our approach differs by arguing that the best/worst cases are closely linked to the possibility of moving first/last.

Our first objective is to align the optimistic and pessimistic approaches with solution concepts in cooperative game theory. Given the main aim of cooperative game theory is distributing profits/costs among agents, we focus on two key solution concepts: the core and the anti-core.

The core (Gillies, 1959) is the set of allocations that distribute the total value while ensuring that each coalition receives at least its intrinsic value. Conversely, the anti-core is constructed by inverting the core inequalities (see, for instance, Oishi et al., 2016).

[1] See Peleg and Sudhölter (2007) for a comprehensive introduction to cooperative games.

In the context of the pessimistic approach, the objective is to guarantee that every coalition attains at least its worst-case scenario, where the total payoff to its members is no less than the lower-bound value on what they collectively generate. On the other hand, the optimistic approach seeks to prevent any coalition from surpassing its best-case scenario, and the corresponding solution concept is the anti-core. This concept is rooted in considerations of fairness (Van Essen and Wooders, 2023) but also serves as a measure of stability, as any coalition exceeding its best-case scenario may cause other agents to withdraw from cooperation.

Once we have defined these best- and worst-case scenarios, it is not immediately evident whether there exists a connection between the objective to ensure the pessimistic lower bounds and the goal of preventing anyone from exceeding the optimistic upper bounds. Our central findings provide the answers: i) the task of guaranteeing allocations that do not exceed the optimistic upper bounds is always at least as challenging as securing allocations surpassing the pessimistic lower bounds, provided that we have defined optimistic and pessimistic bounds in a compatible way, which we explain below, and ii) if externalities are negative, both objectives are always feasible. There is no such guarantee if externalities are positive.

Using a general model accommodating both direct and indirect externalities, we build two families of coalitional games. In the first family, we suppose that a coalition has the initial choice of selecting actions from its feasible options. Our approach allows for versatility with respect to the coalition's objective function. Specifically, we investigate one version where the coalition maximizes the welfare of its own members and another where it maximizes the overall welfare of all players. This dichotomy reflects how we address externalities imposed on others: the former ignores it, as a coalition acts selfishly, while the latter forces the coalition to acknowledge the (direct) externalities imposed on other players. In the second family of games, we shift the perspective, considering a scenario where a coalition is the last to select its actions. In this case, we presume that the other players have already made their optimal choices when selecting first, providing the desired consistency between the two approaches.

We obtain in Theorem 1 the relationships between the anti-core and core of the resulting cooperative games: (i) the anti-core of the game where coalitions make the initial choice is a subset of the core of the game where coalitions select last, and (ii) conversely, the anti-core of the game where coalitions picks last is a subset of the core of the game where coalitions have the first choice.

To make this result meaningful, we transition from choosing first or last to optimistic and pessimistic approaches. This transition becomes particularly straightforward when the sign of the externalities is clear. When externalities are negative, the optimistic approach is to assume that a coalition is a first-mover, while the pessimistic approach suggests the opposite. Conversely, with positive externalities, the optimistic approach is to suppose that a coalition moves last, while the pessimistic approach recommends the opposite. Reinterpreting Theorem 1, we obtain our main contribution: the anti-core of the optimistic game is always a subset of the core of the pessimistic game. Given that in most applications the sign of the externalities is clear, this result is widely applicable. Moreover, if externalities are negative, Theorem 2 shows that the anti-core of the optimistic game is always non-empty. Combining with Theorem 1, the core of the pessimistic game is also non-empty in the case of negative externalities. Essentially, this corollary addresses a central question in the literature, namely, the balancedness of the pessimistic game. It is particularly striking that we are able to obtain this result with very little structure; the result is thus widely applicable. The anti-core of the optimistic game is also more than a tool to ensure the non-emptiness of the core of the pessimistic game; it is, in fact, a refinement of it.

We apply our results to various well-studied applications. These applications build on the links between TU games and joint optimization problems (see, among others, Kalai and Zemel, 1982a; Kalai and Zemel, 1982b; Granot and Granot, 1992). We study applications in which the optimistic and pessimistic approaches are meaningful: queueing theory (Maniquet, 2003; Chun, 2006), minimum cost spanning tree problems (Bergantiños and Vidal-Puga, 2007a), river sharing problems (Ambec and Sprumont, 2002), pipeline externalities problems (Trudeau and Rosenthal, 2023), the location of a facility (Laurent-Lucchetti and Leroux, 2011) and knapsack problems (Arribillaga and Bergantiños, 2022). As our general model encompasses these applications, we reobtain and reinterpret many classic results, while new results also emerge.

In some other applications, optimistic and pessimistic approaches yield dual games. We show the necessary and sufficient condition for this to occur: the sequential and selfish decisions of a coalition picking first and its complement picking last must always yield an optimal outcome for the grand coalition. (Proposition 2). Hence, we establish the coincidence between the anti-core of the pessimistic game and the core of the optimistic game when the games defined on the order of coalitions arrival are dual (Corollary 4). We illustrate duality by means of two well-known applications, bankruptcy (claims) and airport problems (see O'Neill (1982) and Littlechild and Owen (1973) respectively) and show the thin line between duality or no duality with cooperative production problems (Moulin and Shenker, 1992).

The paper is organized as follows. Section 2 provides some preliminaries on TU games. Section 3 introduces the model. We discuss how an optimistic (pessimistic) approach can be interpreted as an upper (lower) bound, and hence the anti-core (core) should be analyzed.

In Section 4 we provide the main result: the relationships between the core and the anti-core of considered games and conclude that the anti-core of the optimistic game is always a subset of the core of the pessimistic game. In Section 5 we apply our model to a wide range of applications that have been well-studied in the literature. In Section 6 we show that, under the presence of duality, the consideration of the two distinct approaches becomes unnecessary. To interpret this, we provide two applications that exhibit duality. Finally, Section 7 concludes.

2. Preliminaries

A *cooperative game with transferable utility* (or TU game) is defined by a pair (N, v) where N is the (finite) set of agents and v is a value function that assigns the value $v(S)$ to each coalition $S \subseteq N$ with $v(\emptyset) = 0$. The number $v(S)$ is the worth of the coalition. Whenever no confusion may arise as to the set of players, we will identify a TU game (N, v) with its value function v.

Given a game v, an allocation is a tuple $x \in \mathbb{R}^N$ representing players' respective allotment. The total payoff of a coalition S is denoted by $x(S) = \sum_{i \in S} x_i$ with $x(\emptyset) = 0$. An allocation is *efficient* if $x(N) = v(N)$, *individually rational* if $x(i) \geq v(\{i\})$ for all $i \in N$, and *coalitionally rational* if $x(S) \geq v(S)$ for all $S \subseteq N$.

An allocation is said to be in the *core* of v if it is efficient and coalitionally rational. Then, the core of the game v is the set of all such allocations: $\mathcal{C}(v) = \left\{x \in \mathbb{R}^N : x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\right\}$. An allocation is said to be in the *anti-core* of v if it is efficient and for all coalitions the reversed coalitional rationality inequalities hold. Then, the anti-core of the game v is the set of all such allocations: $\mathcal{A}(v) = \left\{x \in \mathbb{R}^N : x(S) \leq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\right\}$.

Let $\lambda : 2^N \setminus \{\emptyset\} \to [0, 1]$ where for all $i \in N$ we have $\sum_{S \subseteq N : S \ni i} \lambda_S = 1$. Let Λ be the set of such balanded weights λ. A game is *balanced* if $\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$ for all $\lambda \in \Lambda$. A game v has a non-empty core if and only if it is balanced (Bondareva, 1963; Shapley, 1967). A game is *anti-balanced* if $\sum_{S \subseteq N} \lambda_S v(S) \geq v(N)$ for all $\lambda \in \Lambda$. A game v has a non-empty anti-core if and only if it is anti-balanced.

Convexity and concavity (Shapley, 1971) are conditions that have been extensively studied to prove balancedness. A game (N, v) is said to be *convex* if $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. A game (N, v) is said to be *concave* if $v(T \cup \{i\}) - v(T) \leq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

The Shapley value (Shapley, 1953) is a single-valued solution that has interesting fairness properties. It is the weighted sum of the agents' marginal contributions to all coalitions. Formally, given a game (N, v), the Shapley value $Sh(v)$ assigns to each agent $i \in N$ the payoff $Sh_i(v) = \sum\limits_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} \left[v(S \cup \{i\}) - v(S)\right]$.

A game (N, v^*) is the *dual game* of the game (N, v) if $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

For dual games, it is well-known that the anti-core of v coincides with the core of v^*, and vice versa.

Proposition 1. *If v and v^* are dual, then $\mathcal{A}(v) = \mathcal{C}(v^*)$ and $\mathcal{A}(v^*) = \mathcal{C}(v)$.*

2.1. Optimistic and pessimistic approaches. It is not always trivial to determine what value to assign to a coalition. In the presence of externalities, the value depends on assumptions we make about the behavior of agents external to the coalition considered. Two (opposite) approaches have been extensively studied in the literature. Both take into account how $N \setminus S$ behave depending on the structure of the problem and its associated TU game.

The first approach is the *optimistic approach*. Under the optimistic approach, each coalition is assigned a value corresponding to a best-case scenario which we can interpret as an upper bound on the value coalition S can achieve. Thus, the relevant concept to study is the anti-core under the optimistic approach as it is the set of efficient payoff vectors that assigns to each coalition at most its value.

The second approach is the *pessimistic approach*. Under the pessimistic approach, each coalition is assigned a value corresponding to a worst-case scenario which we can interpret as a lower bound on the value coalition S can achieve. Thus, the relevant concept to study is the core under the pessimistic approach as it is the set of efficient payoff vectors that assigns to each coalition at least as much as its value.

3. The Model

Each agent $i \in N$ can take actions, with the set of possible actions defined as $\mathbb{A}_i$. For each agent, the null action $\emptyset \in \mathbb{A}_i$ means that one possible action is to stay inactive. For each $S \subseteq N$, we define as $\mathbb{A}^S = \bigtimes_{i \in S} \mathbb{A}_i$ the sets of actions jointly available to S and $\mathbb{A} \equiv \mathbb{A}^N$.

For each agent $i \in N$ we have a revenue function $R_i : \mathbb{A}^N \to \mathbb{R}$. Let R represent the set of individual revenue functions. When agents choose their actions, some actions might not be available. We thus define the feasible set, which depends on the actions of other agents. More precisely, for all $S \subseteq N$ and all $a_{N \setminus S} \in \mathbb{A}^{N \setminus S}$, $f_S(a_{N \setminus S}) \subseteq \mathbb{A}^S$ represents the set of actions jointly feasible for S. We suppose that these sets are always non-empty. Since the coalition N includes all players, we write f_N instead of $f_N(\emptyset)$. Let f represent the set of all such feasibility functions for all coalitions S.

The grand coalition faces an optimization problem that we generally write as $\max_{a_N \in f_N} \sum_{i \in N} R_i(a_N)$. We define a problem P as $(\mathbb{A}, f, R)$, which describes the set of actions, the feasibility sets, and the revenue functions. We suppose that problem $P =$

$(\mathbb{A}, f, R)$ has a solution and that for all $i \in N$ we have $R_i(\emptyset_N) = 0$. Let $\mathcal{P}$ be the set of all such problems (for all $\mathbb{A}, f, R$).

Example 1. We use as a running example a simple queueing problem. All agents in N have one single job to be processed on a machine. The machine can process one job per period, and agents have linear waiting costs: if agent i's job is processed in period t, he suffers a cost of $t \times w_i$, where $w_i \geq 0$ is his personal waiting cost parameter.

In this context, we can set $\mathbb{A}_i = \{0, \ldots, |N|\}$ to be theset of periods in which i's job could be processed. Then, for any $S \subseteq N$, $f_S(\emptyset_{N \setminus S})$ represents what is jointly feasible for S if $N \setminus S$ is inactive, i.e., if their jobs are not processed. We then have that $f_S(\emptyset_{N\setminus S})$ is a function $\theta^S : S \to \mathbb{A}^S$ such that $\theta_i^S \neq \theta_j^S$ for all i, j different in S. In words, no two agents in S can be assigned the same processing period.

For $a_{N\setminus S} \neq \emptyset_{N\setminus S}$, we have the additional constraint that $\theta_i^S \neq a_j$ for all $i \in S$ and $j \in N \setminus S$. Stated otherwise, the agents in S cannot be assigned to a period already occupied by an agent in $N \setminus S$.

Finally, we have $R_i(a_N) = -w_i a_i$ for all $i \in N$, i.e., each agent has a disutility w_i per period waiting before the job is processed. We can thus rewrite the problem of the grand coalition as $\max_{\theta\in\Theta(N)} \sum_{i\in N} -\theta_i w_i$ where $\Theta(N)$ is the set of bijections from N to $\{0, \ldots, |N| - 1\}$.

3.1. Externalities. We say that a problem only has **indirect externalities** if for all $i \in N$, all $a_i \in \mathbb{A}_i$ and all $a_{N\setminus i}, a'_{N\setminus i} \in \mathbb{A}^{N\setminus i}$ we have that $R_i\left(\{a_i, a_{N\setminus i}\}\right) = R_i\left(\{a_i, a'_{N\setminus i}\}\right)$. Let $\mathcal{P}^I$ be the set of all such problems.

We say that a problem only has **direct externalities** if for all $S \subseteq N$, and all $a_{N\setminus S}, a'_{N\setminus S} \in \mathbb{A}^{N\setminus S}$ we have that $f_S(a_{N\setminus S}) = f_S(a'_{N\setminus S})$. Let $\mathcal{P}^D$ be the set of all such problems.

We say that a problem exhibits **negative externalities** if for all $i \in S \subseteq N$, all $a_S \in \mathbb{A}^S$, and all $a_{N\setminus S} \in \mathbb{A}^{N\setminus S}$ we have $f_S(a_{N\setminus S}) \subseteq f_S(\emptyset_{N\setminus S})$ and $R_i\left(\{a_S, a_{N\setminus S}\}\right) \leq R_i\left(\{a_S, \emptyset_{N\setminus S}\}\right)$. Let $\mathcal{P}^-$ be the set of all such problems.

We say that a problem exhibits **positive externalities** if for all $i \in S \subseteq N$, all $a_S \in \mathbb{A}^S$, and all $a_{N\setminus S} \in \mathbb{A}^{N\setminus S}$ we have $f_S(a_{N\setminus S}) \supseteq f_S(\emptyset_{N\setminus S})$ and $R_i\left(\{a_S, a_{N\setminus S}\}\right) \geq R_i\left(\{a_S, \emptyset_{N\setminus S}\}\right)$. Let $\mathcal{P}^+$ be the set of all such problems.

Let $\mathcal{P}^e = \mathcal{P}^- \cup \mathcal{P}^+$ be the set of problems with clearly defined externalities.

3.2. Defining cooperative games. We associate cooperative games to problems in $\mathcal{P}$, with the constraint for $P = (\mathbb{A}, f, R)$ that we have $v(N) = \max_{a_N\in f_N} \sum_{i\in N} R_i(a_N)$. For $S \subset N$, we have more flexibility. Because of the externalities, which value to associate to a coalition $S \subset N$ is not clear.

To make sure our proposed methods are clear, we first describe them for the subset of problems with indirect externalities only, before moving to general problems.

3.2.1. *Problems with indirect externalities only.* Because we only have indirect externalities, we have that R_i depends only on a_i. In what follows, we consider the optimal choice for coalition S, for which we will write $R_i(a_S)$ as a shorthand version of $R_i((a_S)_i)$.

We propose two families of functions that allocate values to coalitions. First, we suppose that we compute the value created by coalition S as the maximum value it can create by being the first group to choose their actions. We define such coalition function as[2]

$$v^F(S,P) = \max_{a_S \in f_S(\emptyset_{N\setminus S})} \sum_{i \in S} R_i\left(a_S\right),$$

where $\emptyset_{N\setminus S}$ represent all agents in $N \setminus S$ taking the null action.

We remark that there might be multiple sets of actions for S that maximize the problem yielding $v^F(S,P)$. Which one is picked is irrelevant for the determination of $v^F(S,P)$ but it is important for what follows. Let $\tau(S)$ be an ordering of the set of actions in $\mathbb{A}^S$, $\mathcal{T}(\mathcal{S})$ be the set of all such orderings, and let $\tau = (\tau(S))_{S\in 2^N}$ give us a ranking for all coalitions.

Let $a_{F^\tau(S)}$ be the optimal set of actions taken by S, with τ used as a tiebreaker if needed, such that $\sum_{i\in S} R_i\left(a_{F^\tau(S)}\right) = v^F(S,P)$.

Second, we suppose that coalition S is choosing after $N \setminus S$ has made its own choice. More precisely, we suppose that $N \setminus S$ has optimally chosen its actions when acting first, that S observes those actions (and their effects on its feasible set), before deciding on its own set of actions. When choosing its actions, how $N \setminus S$ has broken ties, if necessary, could impact S, since the externalities are not the same for all maximizers. Thus, we have to define the value obtained by the coalition choosing last as a function of the tiebreaker.

Thus, for any $\tau \in \mathcal{T}$ we have that $v^{L,\tau}(S,P) = \max_{a_S\in f_S(a_{F^\tau(N\setminus S)})} \sum\limits_{i\in N} R_i\left(a_S\right)$. Let $a_{L^\tau(S)}$ be (one of) the optimal set of actions taken by S such that $\sum\limits_{i\in N} R_i\left(a_{L^\tau(S)}\right) = v^{L,\tau}(S,P)$.[3]

Example 2. We revisit our queueing example. v^F supposes that a coalition S has first access to the machine, and that its members can be assigned to any period. It is easy to see that it is optimal to first process the jobs of agents with high waiting costs, that is, for $i,j \in S$, $w_i > w_j \Rightarrow \theta_i < \theta_j$. If $w_i > 0$ for all $i \in S$, then all optimal solutions assign the agents in S to periods 0 to $|S| - 1$. If there are some agents $i \in S$ such that $w_i = 0$ we are indifferent to which period they are assigned to. A natural tiebreaker τ would place these agents directly after the agents with strictly positive waiting cost parameters. Other possible ties (if two or more agents have the same cost parameters) have no impact on the coalitions choosing last, but can be broken by allocating earlier periods to agents identified with a smaller index.

[2]We suppose, in this problem and in subsequent ones, that the optimization problem for coalition $S \subset N$ has a solution.

[3]While there might be multiple maximizers, here it is of no consequence how the tie is broken.

With such a tiebreaker τ, $a_{F(S)}$ is a bijection $\theta^{F,S} : S \to \{0, \ldots, |S| - 1\}$ such that $w_i < w_j \Rightarrow \theta_i^{F,S} < \theta_j^{F,S}$. Then, $v^{L,\tau}$ is such that S considers that the first $|N \setminus S| - 1$ periods are occupied. Thus, we have $a_{L^\tau(S)}$ is a bijection $\theta^{L,S} : S \to \{|N| - |S|, ..., |N|\}$ such that $w_i < w_j \Rightarrow \theta_i^{L,S} < \theta_j^{L,S}$.

3.2.2. *General problems.* For general problems, we still want to define a value function when S picks first, and one when it picks last. However, in the presence of direct externalities, there are additional difficulties.

A first point to consider is that if S chooses first, its actions impose a direct externality on $N \setminus S$. Should these be credited to S, or not? We propose two families of functions that allocate values to coalitions.

We first suppose that coalition $S \subset N$ ignores these externalities and simply maximizes the revenues of its members. We define such coalition function as

$$v^F(S, P) = \max_{a_S \in f_S(\emptyset_{N \setminus S})} \sum_{i \in S} R_i \left(\{a_S, \emptyset_{N \setminus S}\} \right),$$

where $\emptyset_{N \setminus S}$ represent all agents in $N \setminus S$ taking the null action. Let $a_{F^\tau(S)}$ be the optimal set of actions taken by S, with ties broken using τ, such that $\sum_{i \in S} R_i \left(\{a_{F^\tau(S)}, \emptyset_{N \setminus S}\} \right) = v^F(S, P)$.

Alternatively, we also consider the case where $S \subset N$ maximizes the welfare of all agents, which is equivalent to crediting them with the (direct) externalities they impose on others. We define such coalition function as

$$\hat{v}^F(S, P) = \max_{a_S \in f_S(\emptyset_{N \setminus S})} \sum_{i \in N} R_i \left(\{a_S, \emptyset_{N \setminus S}\} \right).$$

Let $\hat{a}_{F^\tau(S)}$ be the optimal set of actions (subject to tiebreaker τ) taken by S such that $\sum_{i \in N} R_i \left(\{\hat{a}_{F^\tau(S)}, \emptyset_{N \setminus S}\} \right) = \hat{v}^F(S, P)$.

We now move to the problem of defining the value credited to S when it picks last. As before, we suppose that S observes the actions chosen by $N \setminus S$. We take the point of view that externalities have to be taken into account somewhere. Thus, we always suppose that when picking last, S takes into account the externalities of its actions on $N \setminus S$. But, if $N \setminus S$ has not taken into account the externalities it has imposed on S, then we will have to add that to the value credited to S.

Once again, our values will depend on the tiebreaker τ. We start with $\hat{v}^{L,\tau}(S)$, which supposes that $N \setminus S$ has first chosen its actions by taking externalities into account. Then,

$\hat{v}^L(S)$ has S to choose its actions to maximize the additional revenues that its actions generate. More precisely:

$$\begin{aligned}\hat{v}^{L,\tau}(S,P) &= \max_{a_S \in f(a_{F^\tau(N\setminus S)})} \sum_{i\in N} \left(R_i\left(\{a_S, a_{F^\tau(N\setminus S)}\}\right) - R_i\left(\{\emptyset_S, a_{F^\tau(N\setminus S)}\}\right)\right) \\ &= \max_{a_S \in f(a_{F^\tau(N\setminus S)})} \sum_{i\in N} R_i\left(\{a_S, a_{F^\tau(N\setminus S)}\}\right) - \hat{v}^F(N\setminus S, P)\end{aligned}$$

Let $\hat{a}_{L^\tau(S)}$ be (one of) the optimal choices for S in the above problem.

Finally, we consider $v^{L,\tau}(S)$, which supposes that $N \setminus S$ has first chosen its actions by ignoring externalities. We still assume that S is credited with the extra value that its actions create on all agents, but to this we now add the externalities imposed by the actions of $N \setminus S$. We thus obtain:

$$\begin{aligned}v^{L,\tau}(S,P) &= \max_{a_S \in f(a_{F^\tau(N\setminus S)})} \sum_{i\in N} \left(R_i\left(\{a_S, a_{F^\tau(N\setminus S)}\}\right) - R_i\left(\{\emptyset_S, a_{F^\tau(N\setminus S)}\}\right)\right) + \sum_{i\in S} R_i\left(\{\emptyset_S, a_{F^\tau(N\setminus S)}\}\right) \\ &= \max_{a_S \in f(a_{F^\tau(N\setminus S)})} \sum_{i\in N} R_i\left(\{a_S, a_{F^\tau(N\setminus S)}\}\right) - \sum_{i\in N\setminus S} R_i\left(\{\emptyset_S, a_{F^\tau(N\setminus S)}\}\right) \\ &= \max_{a_S \in f(a_{F^\tau(N\setminus S)})} \sum_{i\in N} R_i\left(\{a_S, a_{F^\tau(N\setminus S)}\}\right) - v^F(N\setminus S, P)\end{aligned}$$

Let $a_{L^\tau(S)}$ be (one of) the optimal choices for S in the above problem.

Thus, both $\hat{v}^{L,\tau}$ and $v^{L,\tau}$ have the same form: each credits a coalition S with the total value generated when constrained by the initial choice of $N \setminus S$, to which we subtract what was credited to $N \setminus S$.

Going back to the problems with indirect externalities only, we make two remarks. First, because the revenues of an agent depend only on its actions, we have that $v^F = \hat{v}^F$. Second, because the revenues are individually separable, crediting a coalition for the value its initial choice has created simply implies that it maximizes the revenues of its members.

3.3. Optimistic and pessimistic games. When we have negative externalities, it is natural to define the optimistic game as $v^o = v^F$ or $\hat{v}^o = \hat{v}^F$, depending on whether the coalition ignores or does not ignore externalities, and the pessimistic game as the value in the corresponding game where the coalition picks last. Recall that these values might depend on the tiebreaker used. While we could use any tiebreaker, we use the so-called pessimistic tiebreaker τ^p, such that if there is a tie when S picks first, it picks the maximizer that is the worst for $N \setminus S$. Thus, $v^p = v^{L,\tau^p}$ or $\hat{v}^p = \hat{v}^{L,\tau^p}$.

When we have positive externalities, it is natural to define the pessimistic game as $v^p = v^F$ or $\hat{v}^p = \hat{v}^F$. The optimistic game is then defined with the values generated when picking last. We now use the optimistic tiebreaker τ^o, in which the coalition picking first breaks ties by choosing the maximizer that is most favorable for the complement set. Thus, $v^o = v^{L,\tau^o}$ or $\hat{v}^o = \hat{v}^{L,\tau^o}$.

Example 3. In our queueing example, we have negative indirect externalities, as the presence of other players reduces the feasible set for S. Thus, $v^o = v^F$ and $v^p = v^{L,\tau^p}$. Notice that for queueing problems τ^p is what we defined earlier as one of the natural tiebreakers in which agents with $w_i = 0$ are still placed as early as possible, resulting in the assumption that if S picks last, it can only use the last $|S|$ periods.

4. Main results

We now provide our main results. We first establish relationships between the two families of games defined. We first establish links between games where a coalition picks first and games where coalitions pick last, before moving to links between optimistic and pessimistic games.

Theorem 1. *For all $P = (\mathbb{A}, f, R) \in \mathcal{P}$ and all $\tau \in \mathcal{T}$ we have that*

(i) $\mathcal{A}\left(v^F(\cdot, P)\right) \subseteq \mathcal{C}\left(v^{L,\tau}(\cdot, P)\right)$;
(ii) $\mathcal{A}\left(v^{L,\tau}(\cdot, P)\right) \subseteq \mathcal{C}\left(v^F(\cdot, P)\right)$;
(iii) $\mathcal{A}\left(\hat{v}^F(\cdot, P)\right) \subseteq \mathcal{C}\left(\hat{v}^{L,\tau}(\cdot, P)\right)$;
(iv) $\mathcal{A}\left(\hat{v}^{L,\tau}(\cdot, P)\right) \subseteq \mathcal{C}\left(\hat{v}^F(\cdot, P)\right)$.

Proof. We start with part (**i**). Fix $\tau \in \mathcal{T}$. Fix P, and thus write $v^F(S)$ and $v^{L,\tau}(S)$ instead of $v^F(S, P)$ and $v^{L,\tau}(S, P)$. Let $v(N) \equiv \max_{a_N \in f_N} \sum_{i \in N} R_i(a^N)$. Notice that $v^F(N) = v^{L,\tau}(N) = v(N)$.

An allocation $x \in \mathcal{A}\left(v^F\right)$ if $v(N) - v^F(N \setminus S) \leq x(S) \leq v^F(S)$ for all S. An allocation $x \in \mathcal{C}\left(v^{L,\tau}\right)$ if $v^{L,\tau}(S) \leq x(S) \leq v(N) - v^{L,\tau}(N \setminus S)$. It is easy to see that $\mathcal{A}\left(v^F\right) \subseteq \mathcal{C}\left(v^{L,\tau}\right)$ if and only if $v(N) \geq v^F(S) + v^{L,\tau}(N \setminus S)$ for all S. Fix S and let a_N^* be (one of) the optimal set(s) of actions taken by N. Then, $v(N) = \sum_{i \in N} R_i(a_N^*)$, $v^F(S) = \sum_{i \in S} R_i\left(\{a_{F^\tau(S)}, \emptyset_{N \setminus S}\}\right)$ and $v^L(N \setminus S) = \sum_{i \in N} R_i\left(\{a_{F^\tau(S)}, a_{L^\tau(N \setminus S)}\}\right) - v^F(S)$. We thus have that

$$\begin{aligned} v^F(S) + v^L(N \setminus S) &= v^F(S) + \sum_{i \in N} R_i\left(\{a_{F^\tau(S)}, a_{L^\tau(N \setminus S)}\}\right) - v^F(S) \\ &= \sum_{i \in N} R_i\left(\{a_{F^\tau(S)}, a_{L^\tau(N \setminus S)}\}\right) \\ &\leq \sum_{i \in N} R_i(a_N^*) \\ &= v(N), \end{aligned}$$

where the inequality follows from the fact that, by definition, $(a_{F^\tau(S)}, a_{L^\tau(N \setminus S)}) \in f_N$.

Next, we show (**ii**). An allocation $x \in \mathcal{A}\left(v^{L,\tau}\right)$ if $v(N) - v^{L,\tau}(N \setminus S) \leq x(S) \leq v^{L,\tau}(S)$ for all S. An allocation $x \in \mathcal{C}\left(v^F\right)$ if $v^F(S) \leq x(S) \leq v(N) - v^F(N \setminus S)$. It is easy to see that $\mathcal{A}\left(v^{L,\tau}\right) \subseteq \mathcal{C}\left(v^F\right)$ if and only if $v(N) \geq v^{L,\tau}(S) + v^F(N \setminus S)$ for all S, which we have already shown in the proof of part (**i**). The proofs of parts (**iii**) and (**iv**) are identical. □

Recall that when we have negative externalities, $v^o = v^F$ or $\hat{v}^o = \hat{v}^F$, and $v^p = v^{L,\tau^p}$ or $\hat{v}^p = \hat{v}^{L,\tau^p}$. Moreover, when we have positive externalities, $v^o = v^{L,\tau^o}$ or $\hat{v}^o = \hat{v}^{L,\tau^o}$, and $v^p = v^F$ or $\hat{v}^p = \hat{v}^F$. Hence, we have the following corollary to Theorem 1, which states that whenever the sign of the externalities are clear, the anti-core of the optimistic game is a subset of the core of the pessimistic game.

Corollary 1 (of Theorem 1)**.** *For all $P \in \mathcal{P}^e$ we have that $\mathcal{A}\left(v^o(\cdot, P)\right) \subseteq \mathcal{C}\left(v^p(\cdot, P)\right)$ and $\mathcal{A}\left(\hat{v}^o(\cdot, P)\right) \subseteq \mathcal{C}\left(\hat{v}^p(\cdot, P)\right)$.*

Proof. We show that $\mathcal{A}\left(v^o\right) \subseteq \mathcal{C}\left(v^p\right)$. If $P \in \mathcal{P}^-$, then $v^o = v^F$ and $v^p = v^{L,\tau^p}$. The result follows from the first part of Theorem 1. If $P \in \mathcal{P}^+$, then $v^o = v^{L,\tau^o}$ and $v^p = v^F$. It follows from the second part of Theorem 1.

The proof for $\mathcal{A}\left(\hat{v}^o\right) \subseteq \mathcal{C}\left(\hat{v}^p\right)$ is identical. □

For the remainder of the paper, given that the results do not depend on the chosen tiebreaker, we do not explicitly mention them.

Next, we show that when we have negative externalities the anti-core of the optimistic game is always non-empty.

Theorem 2. *For all $P \in \mathcal{P}^-$, $\mathcal{A}\left(v^o(\cdot, P)\right)$ is non-empty.*

Proof. Fix $P \in \mathcal{P}^-$ and write $v^F(S)$ and $v^o(S)$ instead of $v^F(S, P)$ and $v^o(S, P)$. Since $P \in \mathcal{P}^-$ we have that $v^o = v^F$.

Let a^* be (one of) the maximizer(s) for the problem of the grand coalition. We first show that $v^o(S) \geq \sum_{i \in S} R_i\left(a^*\right)$.

We have that

$$
\begin{aligned}
v^o(S) &= \sum_{i \in S} R_i\left(\left\{a_{F(S)}, \emptyset_{N \setminus S}\right\}\right) \\
&\geq \sum_{i \in S} R_i\left(\left\{a^*_S, \emptyset_{N \setminus S}\right\}\right) \\
&\geq \sum_{i \in S} R_i\left(\left\{a^*_S, a^*_{N \setminus S}\right\}\right) \\
&= \sum_{i \in S} R_i\left(a^*\right),
\end{aligned}
$$

where the first inequality is because, as we have negative externalities, $a_S^* \in f_S\left(a_{N\setminus S}^*\right) \subseteq f_S\left(\emptyset_{N\setminus S}\right)$ and the second, once again as a consequence of negative externalities, because $R_i\left(\{a_S^*, \emptyset_{N\setminus S}\}\right) \geq R_i\left(\{a_S^*, a_{N\setminus S}^*\}\right)$ for all $i \in S$.

Then, take $\lambda \in \Lambda$ and multiply the inequality by λ_S and sum over S to obtain

$$\begin{aligned}
\sum_{S\subset N} \lambda_S v^o(S) &= \sum_{S\subset N} \lambda_S \sum_{i\in S} R_i\left(\{a_{F(S)}, \emptyset_{N\setminus S}\}\right) \\
&\geq \sum_{S\subset N} \lambda_S \sum_{i\in S} R_i\left(a^*\right) \\
&= \sum_{i\in N} \sum_{S\ni i} \lambda_S R_i\left(a^*\right) \\
&= \sum_{i\in N} R_i\left(a^*\right) \\
&= v^o(N).
\end{aligned}$$

Thus, v^F is anti-balanced and its anti-core is non-empty. □

Combining our two main results, we obtain the following corollary.

Corollary 2. *For all* $P \in \mathcal{P}^-, \emptyset \neq \mathcal{A}\left(v^o(\cdot, P)\right) \subseteq \mathcal{C}\left(v^p(\cdot, P)\right)$.

Thus, with very little structure on the problem other than negative externalities, we are able to show to non-emptiness of the pessimistic core.

The guarantee of a non-empty anti-core does not carry over to problems with positive externalities, as illustrated in the following counterexample.

Example 4. Atay and Trudeau (2024) provide a variant of the queueing problem by supposing that agents must buy machines to queue on, and can buy as many machines as they want. The problem becomes one with positive direct externalities: by itself, a coalition can only buy its own machines and queue on them; if it joins others, it can still do so, but can also take advantage of unused time slots on their machines. Atay and Trudeau (2024) show that the core of the corresponding pessimistic game is sometimes empty, sometimes not. Given Theorem 1, so is the anti-core of the optimistic game.

5. Applications

Our model allows for both direct and indirect externalities. In practice, most applications feature only one of the two types. In turn, we discuss several applications. We examine how optimistic and pessimistic approaches have been defined in each case, how that translate into our framework and if our results allow to reinterpret existing results or obtain new ones. We begin by examining applications with indirect externalities, followed by those with direct externalities, and conclude with applications involving both types of externalities. Table 1 below summarizes the applications and how they fit in our framework.

		direct externalities		
		+	0	-
indirect externalities	+		mcst, river sharing (ATS doctrine)	
	0	locating facility		pipeline extern.
	-	knapsack	queueing, river sharing (UTI doctrine)	

TABLE 1. Classification of applications and externalities in our framework

5.1. Indirect externalities. A problem $P \in \mathcal{P}$ exhibits indirect externalities when in the optimization problem the presence of other agents does not directly impact the optimizing agent's payoff but affects its feasible set. Recall that a problem only has indirect externalities if for all $i \in N$, all $a_i \in \mathbb{A}_i$ and all $a_{N\setminus i}, a'_{N\setminus i} \in \mathbb{A}^{N\setminus i}$ we have that $R_i\left(\{a_i, a_{N\setminus i}\}\right) = R_i\left(\{a_i, a'_{N\setminus i}\}\right)$.

Next, we explore two applications that exhibit indirect externalities to illustrate how our model applies and can offer new insights into established results.

5.1.1. *Queueing problems.* We first examine more formally our running example of queueing problems. Consider a set of agents N that each have a job to be processed at one machine. The machine can process only one job at a time. Each agent $i \in N$ incurs waiting costs $w_i > 0$ per unit of time. The queueing problem determines both the order in which to serve agents and the corresponding monetary transfers they should receive (see Chun (2016) for a survey on the queueing problem).

It can be solved by taking various approaches including TU games. Then, Maniquet (2003) concentrates on the fairness aspect of the problem under the assumption that a coalition is served before the players outside the coalition. The minimal transfer rule[4], ϕ^{min}, is obtained by applying the Shapley value to v^o, the game under this optimistic assumption.

[4]The minimal transfer rule assigns to each agent a position in the queue and a monetary transfer. The monetary transfer is equal to half of their unit waiting cost multiplied by the number of agents in front of them in the queue subtracted by half of the sum of the unit waiting costs of the people behind them in the queue.

Alternatively, Chun (2006) assumes that a coalition is served after the non-coalitional members. The maximal transfer rule[5], ϕ^{max}, is obtained by applying the Shapley value to v^p, the game under this pessimistic assumption.

In our framework, $R_i = -w_i r_i(\sigma)$ where $r_i(\sigma)$ is the rank of agent i in the queue σ. Since a machine cannot serve more than a job at a given time, a queue σ is feasible for any coalition if there are no $i, j \in N$ such that $r_i(\sigma) = r_j(\sigma)$. Then, $f_S(a_{F(N\setminus S)})$ is the feasible set for S when the coalition $N \setminus S$ takes the first $|N \setminus S|$ positions in the queue. Hence, this problem exhibits negative indirect externalities.

It has been shown that v^o is concave and v^p is convex, resulting in their Shapley values being respectively in the anti-core of v^o and the core of v^p. We thus obtain the following results.

Theorem 3. *For any queueing problem, we have $\phi^{min} \in \mathcal{A}(v^o) \subseteq \mathcal{C}(v^p)$ and $\phi^{max} \in \mathcal{C}(v^p)$.*

The results that $\{\phi^{min}, \phi^{max}\} \in \mathcal{C}(v^p)$ come respectively from Maniquet (2003) and Chun (2006). We obtain an additional justification for the minimal transfer rule, as it offers allocations that are below the optimistic bounds and above the pessimistic bounds. The maximal transfer rule offers allocations above the pessimistic bounds, but not always below the optimistic bounds.

5.1.2. *Minimum cost spanning tree problems.* We have a set of nodes consisting of $N_0 \equiv N \cup \{0\}$, where 0 is a special node called the source. Agents need to be connected to the source to obtain a good or a service. To each edge $(i, j) \in N_0 \times N_0$ corresponds a cost $c_{ij} \geq 0$, with the assumption that $c_{ij} = c_{ji}$. These costs are fixed costs, paid once if an edge is used, regardless of how many agents use it. The problem is to connect all agents to the source at the cheapest cost. Given the assumptions above, among the optimal networks there always exists a spanning tree, hence the name of the problem. A minimum cost spanning tree (mcst) problem is (N, c), where c is the list of all edge costs. c is also called a cost matrix.

In our framework, $R_i = -c_{p(i)i}$, where $p(i) \in N_0$ is the predecessor of i in the unique path from 0 to i in the spanning tree. The usual assumption is to suppose that a coalition S cannot use edge (i, j) if $i, j \in N_0 \setminus S$. Then $f_S(\emptyset)$ is the set of spanning trees rooted at 0, while $f_S(a_{N\setminus S})$, for any $a_{N\setminus S}$ such that $a_i^{N\setminus S} \neq \emptyset$ for all $i \in N \setminus S$ also treats the nodes in $N \setminus S$ as sources. Thus, we obtain a problem with positive indirect externalities.

Most of the literature has considered the pessimistic game v^p, in which a coalition S connects to the source first, before $N \setminus S$. It corresponds here to v^F. An exception is Bergantiños and Vidal-Puga (2007b), which considers the optimistic game v^o, in which

[5]The maximal transfer rule assigns to each agent a position in the queue and a monetary transfer. The monetary transfer is equal to a half of the sum of the unit waiting costs of her predecessors minus a half of her unit waiting cost multiplied by the number of her followers.

coalition S supposes that $N \setminus S$ has already connected to the source. It corresponds, in our notation, to v^L.

The literature has devoted considerable attention to the notion of irreducible cost matrix (Feltkamp et al., 1994; Bergantiños and Vidal-Puga, 2007a): since many edges are not used in any optimal spanning tree, we reduce the cost of these edges as much as possible, under the constraint that $v^p(N)$ does not change. There is a unique way to do so, and irreducible edge costs can be obtained as follows: take any optimal spanning tree, and for each pair of nodes $(i,j) \in N_0$, look at the (unique) path from one to the other, and assign to (i,j) the most expensive edge on that path. We then obtain the irreducible cost matrix $\bar{c}$. Let $\bar{v}^p$ and $\bar{v}^o$ be the pessimistic and optimistic games obtained from the irreducible cost matrix.

Theorem 4. *(Bergantiños and Vidal-Puga, 2007b). For any mcst problem (N,c) we have*

(i) *$\bar{v}^p$ and $\bar{v}^o$ are dual.*

(ii) *$\bar{v}^o = v^o$.*

This leads us, using our results, to the following corollary.

Corollary 3. *For any mcst problem (N,c) we have $\mathcal{A}(v^o) = \mathcal{C}(\bar{v}^p)$.*

This result is interesting for two reasons. First, $\mathcal{C}(\bar{v}^p)$ is called the irreducible core (Bird, 1976), and has been shown to be uniquely characterized by additivity and monotonicity properties (Tijs et al., 2006; Bergantiños and Vidal-Puga, 2015). Second, our equivalence with the anti-core of the optimistic game means that we do not need to go through the modification of the cost matrix into the irreducible matrix to obtain the irreducible core.

5.1.3. *River sharing problems.* Suppose a river described as a line with agents i being upstream of agent j if and only $i < j$. There is an entry $e_i \geq 0$ of water at each location i, and the water that flows at location i can be consumed by agent i or allowed to flow downstream. The benefit from water consumption for agent i is given by a strictly increasing and strictly concave function b_i. A water sharing problem is (N, e, b), the set of players, the vector of water entries, and the collection of benefit functions (Ambec and Sprumont, 2002). The problem for the grand coalition is to maximize joint benefits, under the constraint imposed by the flows of water. If $x_i \geq 0$ is the consumption level of agent i, we have, for any $i \in N$, that $\sum_{j \leq i} x_i \leq \sum_{j \leq i} e_i$.

Various coalitional value functions have been defined for the problem, interestingly constructed from various doctrines used in international law. Under the unlimited territorial integrity (UTI) doctrine, an agent can consume any water that flows through its location. Combined with the assumption that coalition S acts before $N \setminus S$, we obtain that $v^{o,UTI}(S) = \max_{(x_i)_{i \in S}} \sum_{i \in S} b_i(x_i)$ under the constraints that $\sum_{\substack{j \leq i \\ j \in S}} x_i \leq \sum_{j \leq i} e_i$ for all $i \in S$.

Let $x^{o,UTI}(S)$ be the optimal consumption for S in that context.[6] Given the optimistic nature of the problem, $v^{o,UTI}(S)$ is seen as an upper bound on the welfare of S.

Under the absolute territorial integrity (ATS) doctrine, an agent has absolute right over the water entry on its territory. Thus, while we still suppose that coalition S acts before $N \setminus S$, it affects the feasibility set: if $i, j \in S$ are non-consecutive on the river, S anticipates that any attempt to transfer water from i to j will fail, as that water will be consumed by adjoining agents. To properly define the constraints in this new feasible set, we say that a coalition is consecutive if for any pair of agents in that coalition, adjoining agents are also in the coalition. Let $\Gamma(S)$ be the coarsest partition of S into consecutive coalitions. Then, $v^{o,ATS}(S) = \sum_{T\in\Gamma(S)} \max_{(x_i)_{i\in T}} \sum_{i\in T} b_i(x_i)$ under the constraints that $\sum_{\substack{j\leq i\\ j\in T}} x_i \leq \sum_{\substack{j\leq i\\ j\in T}} e_i$ for all $i \in T$ and all $T \in \Gamma(S)$. Let $x^{o,ATS}(S)$ be the optimal consumption for S in that context. Notice that given the restrictive feasibility set that forces a consecutive coalition to consume the water of its members, it does not matter if a coalition picks first or last, resulting in $v^{o,ATS}(S) = v^{p,ATS}(S)$. An immediate consequence is that typically $\mathcal{A}(v^{o,ATS})$ is empty. This is not in contradiction with Theorem 2, as under the ATS doctrine the problem now has positive externalities, as cooperation allows to form bigger consecutive coalitions that can improve the allocation of water.

Given the pessimistic constraints in the ATS version of the problem, $v^{p,ATS}(S)$ is seen as a lower bound on the welfare of S.

While the two doctrines yield lower and upper bounds and what is described as optimistic and pessimistic approaches, the relationship between the two games cannot be obtained via our Theorem 1, which would require that in one game S acts after $N \setminus S$ has chosen its optimal actions as in the other game. Here, we can complement game $v^{o,UTI}$ with a game that we call $v^{p,UTI}$, defined as follows: a coalition S determines its consumption levels after coalition $N \setminus S$ has chosen $x^{o,UTI}(N \setminus S)$. This implies that only coalitions of consecutive agents containing agent n can consume any amount of water. Thus, we have that $v^{p,UTI}(S) = 0$ if $n \notin S$ and $v^{p,UTI}(S) = v^{p,ATS}(S^n)$ otherwise, where S^n is the largest consecutive coalition in S that contains n. It is thus easy to see that $v^{p,UTI} \leq v^{p,ATS}$, and thus that $\mathcal{C}(v^{p,ATS}) \subseteq \mathcal{C}(v^{p,UTI})$.

Ambec and Sprumont (2002) define the downstream incremental allocation rule, which is such that $y_i^{DI} = v^{o,UTI}(\{1,\ldots,i\}) - v^{o,UTI}(\{1,\ldots,i-1\}) = v^{p,ATS}(\{1,\ldots,i\}) - v^{p,ATS}(\{1,\ldots,i-1\})$. We have the following results.

Theorem 5. *For all river sharing problem (N, e, b), we have:*

- ***(i)*** *(Ambec and Sprumont, 2002): $y^{DI} = \mathcal{A}(v^{o,UTI}) \cap \mathcal{C}(v^{p,ATS})$;*
- ***(ii)*** *$y^{DI} \in \mathcal{C}(v^{p,UTI})$.*

[6] Under the given assumptions, the maximizer is unique.

Part (**ii**) is a simple consequence of our main theorem.

Many extensions of the model have been considered, including to cases where some agents can be satiated (Ambec and Ehlers, 2008) and cases with multiple springs and bifurcations (Khmelnitskaya, 2010). See Béal et al. (2012) for a review.

5.2. Direct externalities. A problem $P \in \mathcal{P}$ exhibits direct externalities when in the optimization problem the action chosen by other agents directly impacts the optimizing coalition's payoff but does not affect its feasible set. Recall that a problem only has direct externalities if for all $S \subseteq N$, and all $a_{N\setminus S}, a'_{N\setminus S} \in \mathbb{A}^{N\setminus S}$ we have that $f_S(a_{N\setminus S}) = f_S(a'_{N\setminus S})$.

Next, we explore applications that exhibit direct externalities to illustrate how our model applies and can offer new insights into established results.

5.2.1. *Pipeline externalities problems.* Trudeau and Rosenthal (2023) propose the *pipeline externalities problem*, which is closely related to the river sharing problem. Agents are also located on a line, now modeling a pipeline. Upstream of agent 1 is the source of a desirable good (e.g., water, oil, natural gas). The delivery of this good to an agent i however creates local damages (e.g., pollution, congestion) to the agents upstream. The assumption is that the marginal benefit of consumption is non-increasing and the marginal damage is non-decreasing in the flow at each location. The model encompasses the river sharing problem if we reverse the flow: in river sharing problem the consumption of upstream agents reduces the potential consumption of agents downstream, while in the pipeline externalities model the consumption of agents downstream creates negative local externalities upstream. While the pipeline externalities model can accommodate any convex damage functions, the river sharing problem and its feasibility constraints are equivalent to a very specific damage function: the damage is zero until we hit the feasibility constraint at a given location, and infinite afterwards. The pipeline externalities model is thus better seen as one with negative direct externalities.

More precisely, Trudeau and Rosenthal (2023) suppose that consumption and flows are in discrete units. To each agent $i \in N$, a vector u^i of non-increasing marginal utilities and a vector d^i of non-decreasing marginal damages are associated, resulting in a problem (N, u, d). The problem of the grand coalition is $\max_{z\in\mathbb{R}^N_+} \sum_{i\in N} \left(\sum_{k=1}^{z_i - z_{i-1}} u^i_k - \sum_{k=1}^{z_i} d^i_k\right)$. Trudeau and Rosenthal (2023) study $\hat{v}^o$, the game in which a coalition S chooses first its optimal consumption, but considers the (negative) impact its consumption has on other agents. This is equivalent to the case in which coalition S can use the pipeline as it wishes, conditional on compensating other agents for the negative externalities it generates. The complement game $\hat{v}^p$ is such that S comes in after $N \setminus S$ has optimally used the pipeline to its linking, subject to compensations to other agents. Trudeau and Rosenthal (2023) show that while $\hat{v}^p$ is not convex, $\hat{v}^o$ is concave, and thus they obtain that its Shapley value is in the anti-core of $\hat{v}^o$.

Theorem 6. *(Trudeau and Rosenthal, 2023) For any pipeline externalities problem (N, u, d), $Sh(\hat{v}^o) \in \mathcal{A}(\hat{v}^o) \subseteq \mathcal{C}(\hat{v}^p)$.*

This complements our Theorem 2 that allows us to conclude that $\emptyset \neq \mathcal{A}(v^o) \subseteq \mathcal{C}(v^p)$.

5.2.2. *Locating a socially desirable but locally undesirable facility.* Consider the problem of locating a socially desirable but locally undesirable facility, such as a garbage dump or a jail, in one of N communities (Laurent-Lucchetti and Leroux, 2011). While each community $i \in N$ receives a benefit $b_i \geq 0$ if the facility is built, it also prefers that it not be located in their community, which results in a cost c_i. We suppose that communities are ordered in non-decreasing order of their hosting cost, and we suppose that $\sum_{i=1}^{n} b_i > c_1$, so that it is always optimal to build the facility in community 1. A problem is (N, b, c), where b and c are respectively the vectors of benefits and hosting costs. This is a problem with positive direct externalities.

A pessimistic approach would suppose that community S would have to build the facility in one of its communities, if the benefits of its members warrant it. This corresponds, in our notation, to v^F. In the optimistic approach, S takes into account what $N \setminus S$ has chosen in the pessimistic approach. If $N \setminus S$ has built the facility, S simply free rides on the location and the value it adds is the sum of the benefits enjoyed by its members. If $N \setminus S$ has decided not to build the facility, then S can either build it and locate it in one of its communities with the cheapest hosting cost, or decide not to build it, if $\sum_{i=1}^{n} b_i - c_i < 0$ for all $i \in S$. This corresponds, in our notation, to v^L.[7]

Laurent-Lucchetti and Leroux (2010) proposes an allocation based on the Lindahl prices: agent i pays a fraction $\frac{b_i}{B}$ of the hosting costs, that is transferred to the host, with $B = \sum_{i=1}^{n} b_i$. Then, for all $i \in N$, the final allocation of net benefit generated by the facility is $y_i^{LINDAHL} = \frac{b_i(B-c_1)}{B}$.

Theorem 7. *For all problems (N, b, c), $y^{LINDAHL} \in \mathcal{C}(v^p)$. There are problems (N, b, c) where $y^{LINDAHL} \notin \mathcal{A}(v^o)$.*

Proof. To prove the first part, fix a problem (N, b, c) and $S \subset N$. We have that $\sum_{i \in S} y_i^{LINDAHL} = \frac{b(S)(B-c_1)}{B}$ where $b(S) = \sum_{i \in S} b_i$, and $v^p(S) = \max\{0, b(S) - \min_{i \in S} c_i\}$.

By construction, $\sum_{i \in S} y_i^{LINDAHL} \geq 0$, so we are left to check that $\sum_{i \in S} y_i^{LINDAHL} \geq b(S) - \min_{i \in S} c_i$. But it is true if $\frac{b(S)(B-c_1)}{B} = b(S) - \frac{b(S)c_1}{B} \geq b(S) - \min_{i \in S} c_i$, which simplifies to $\min_{i \in S} c_i \geq \frac{b(S)c_1}{B}$. But we have that $\min_{i \in S} c_i \geq c_1 \geq \frac{b(S)c_1}{B}$, as $\frac{b(S)}{B} \leq 1$, and thus it is verified.

[7] $\hat{v}^F$ is well-defined here but seems less natural: In that pessimistic game a coalition builds the facility in one of its communities if the sum of benefits, for all agents in N, is greater than its cheapest hosting cost.

The second part is proved by constructing a counterexample. Suppose that $N = \{1, 2\}, b = (6, 12)$ and $c = (15, 20)$. It is then optimal to locate the facility in community 1, generating a net benefit of 3. We have that $y^{LINDAHL} = (1, 2)$. But consider $v^o(\{2\})$. When first, community 1 has not built the facility. Community 2, arriving last, can then either construct the facility at its location and get credit for the net benefit, but since it is -2, it prefers not to build it, and thus $v^o(\{2\}) = 0$. Since $y_2^{LINDAHL} = 2 > 0 = v^o(\{2\})$, $y^{LINDAHL} \notin \mathcal{A}(v^o)$. □

5.3. Combined externalities. While the majority of applications typically have one type of externality, there are some applications that have both types. In this subsection, we present an application that showcases both direct and indirect externalities to illustrate how our model applies and can offer new insights into established results.

5.3.1. *Knapsack problems.* In a knapsack problem, agents have objects with different valuations, while the knapsack has a fixed capacity. Hence, they face an optimization problem that maximizes the value of the objects carried while respecting the capacity of the knapsack. There is a set of agents N that decide which objects in M should be placed in a fixed size knapsack Q. Each object $j \in M$ has a fixed size $q_j \in \mathbb{R}_+$. Then, we can define the revenue of an agent $i \in N$ when a unit of object $j \in M$ is added to the knapsack by r_j^i. We denote a knapsack problem by (N, M, Q, q, r). In the literature, it is assumed that each individual considers an optimization problem to maximize the social welfare (see for instance Kellerer et al., 2004). Based on this, Arribillaga and Bergantiños (2022) considers three approaches. In their optimistic game, it is assumed that the knapsack is filled with objects in the best way for a coalition S while in their realistic game the knapsack is filled with objects in the best way for the coalition $N \setminus S$ and then coalition S maximizes its revenue. In a third approach, which they call the pessimistic game, the agents in $N \setminus S$ aim to hurt the agents in S by filling the knapsack in a way that would be most harmful to coalition S.

In our framework, we observe both direct (objects in the knapsack are public goods) and indirect externalities (whatever object selected by others reduces the available room in the knapsack). Consider a coalition S. If it chooses first, then its feasible set is larger than when it chooses after $N \setminus S$ since it can fill the knapsack with more objects if it chooses first. That is, $f_S(\emptyset_{N\setminus S}) \supseteq f_S(a_{F(N\setminus S)})$. Besides, if coalition S chooses after $N \setminus S$, since objects have a public good nature, the action of $N \setminus S$ also increases the revenue of S. Nonetheless, unless $N \setminus S$ has a limited set of objects that generate positive revenue, it fills the knapsack up to its capacity. In turn, coalition S will not be able to add any objects if it chooses after $N \setminus S$. Hence, the optimistic approach assumes that coalition S chooses first and maximizes its own welfare, obtaining v^F with action $a_{F(S)}$. Meanwhile, the realistic game of Arribillaga and Bergantiños (2022) considers that $N \setminus S$ has filled the knapsack selfishly, before allowing coalition S to add objects, if any room is left. This, in our context, is equivalent to v^L. While here we have indirect and direct externalities of opposite signs, it is easy to verify that for all $S \subseteq N$, $v^F(S) \geq v^L(S)$. Thus, the optimistic approach consists in choosing first and the pessimistic approach in choosing last.

We use a simple allocation rule, that we call the no-transfer rule, and show that it belongs to the anti-core of the optimistic game, and thus the core of the pessimistic game. Let $M^* \subseteq M$ be (one of) the optimal subset(s) of objects that maximize the welfare of the grand coalition. Then, we simply allocate to each agent the revenue obtained from these objects: $x_i^{nt} = \sum_{k \in M^*} r_k^i$ for all $i \in N$.

Theorem 8. *For any knapsack problem (N, M, Q, q, r), $x^{nt} \in \mathcal{A}(v^o) \subseteq \mathcal{C}(v^p)$.*

The result easily follows from this observation: selecting M^* is feasible for coalition S when they choose first (which is how we obtain their optimistic value). If they pick anything else, it is because that gives them higher joint revenues. Thus, $\sum_{i \in S} x_i^{nt} \leq v^o(S)$ as desired.

Arribillaga and Bergantiños (2022) also describe their own pessimistic version of the game, which is much more pessimistic than what was described above: they suppose that not only coalition S fills the knapsack after $N \setminus S$, but that $N \setminus S$ has purposefully filled the knapsack with the worst combination for S. This is in contrast to our approach, that cannot embrace an objective that does not consist of a coalition maximizing its objective welfare (and possibly the welfare of others). It also leads to a disjoint jump between the objective functions of coalitions S and N, as that latter coalition must always maximize its joint benefits. That being said, given that this "extremely pessimistic" approach will lead to lower values, its core will be a superset of the core of our pessimistic game.

Table 2 summarizes the concepts of action, feasible set, and revenue function for the applications studied in this section.

	Actions	**Feasible sets**	**Revenue functions**
Queueing	Time-slots at which you are served	Not two agents can select the same time-slot	Waiting cost to be served in given time-slot
Mcst	Agents (or source) you can directly connect to	Set of connections must form a mcst rooted at the source	Cost to build edge to the agent you directly connect to
River sharing	Amount of water consumed	Must satisfy the feasibility constraint imposed by the flow (depends on doctrine)	Utility of water consumed
Pipeline externalities	Amount of good consumed	No additional constraint	Utility of good consumed net of externalities imposed by the flow at your location
Locating facility	Host or not the facility	No additional constraint	Utility of facility if someone hosts, minus hosting cost if hosting. Zero if nobody hosts.
Knapsack	Goods that can be carried in knapsack	Group must collectively choose; goods must fit in knapsack	Utility of the chosen goods

TABLE 2. Primitives of the model for applications considered in Section 5.

6. Duality

In this section we study the duality between the games defined in Section 3.

Following Proposition 1, it is unnecessary to define optimistic and pessimistic games when the games v^F and v^L (or $\hat{v}^o$ and $\hat{v}^p$) are dual.

In our framework, v^F and v^L are dual if and only if for any coalition S, letting S pick first and $N \setminus S$ react to that afterward always leads to an efficient outcome. In other words, an optimal outcome can always be obtained by sequential selfish optimizations by S and $N \setminus S$. A similar result is obtained for $\hat{v}^F$ and $\hat{v}^L$.

Proposition 2. *For a problem $P = (\mathbb{A}, f, R) \in \mathcal{P}$, we have that*

(i) *v^F and v^L are dual if and only if for all $S \subset N$ there exists $a_{F(S)}$ and $a_{L(N\setminus S)}$ such that $\{a_{F(S)}, a_{L(N\setminus S)}\}$ is a maximizer of $\max_{a_N \in f_N} \sum_{i\in N} R_i(a_N)$.*

(ii) *$\hat{v}^F$ and $\hat{v}^L$ are dual if and only if for all $S \subset N$ there exists $\hat{a}_{F(S)}$ and $\hat{a}_{L(N\setminus S)}$ such that $\{\hat{a}_{F(S)}, \hat{a}_{L(N\setminus S)}\}$ is a maximizer of $\max_{a_N \in f_N} \sum_{i\in N} R_i(a_N)$.*

Corollary 4 (of Proposition 1)**.** *For all $P \in \mathcal{P}^e$, if v^o and v^p are dual, then $\mathcal{A}(v^o) = \mathcal{C}(v^p)$, and if $\hat{v}^o$ and $\hat{v}^p$ are dual, then $\mathcal{A}(\hat{v}^o) = \mathcal{C}(\hat{v}^p)$.*

6.1. Applications with duality. Two important applications exhibiting duality are bankruptcy (claims) problems and airport problems.

The *bankruptcy problem* deals with sharing an estate E of a perfectly divisible resource among agents N who have conflicting claims. That is, the sum of claims is larger than the estate: $\sum_{i\in N} c_i > E$ where c_i is the claim of agent i. O'Neill (1982) studied such problems from an economic point of view. He introduced an associated TU game to each bankruptcy problem and also defined the run-to-the-bank rule based on an average over all possible orders on agents arrival.

The optimistic approach would correspond to a bank-run situation, in which coalition S arrives first and collects its combined claim or the endowment, whichever is smallest. The pessimistic approach has coalition S arriving last, collecting what is left after the bank run of $N \setminus S$. The combination of the optimistic action of S and the pessimistic action of $N \setminus S$ always leads to a full distribution of the endowment, and thus to an efficient outcome. Following Proposition 2, the two games are dual.

The *airport problem* introduced by Littlechild and Owen (1973) aims to allocate the cost of a landing strip among users with varying runway length requirements. Every agent i requires a length l_i at the runway. It is assumed that the cost to build the runway is non-decreasing in its length. That is, for any two agents i and j such that $l_i < l_j$, $c(l_i) \leq c(l_j)$.

First, take the pessimistic approach and assume that coalition S arrives first to build its runway. The longest runway required by a member of the coalition will be built, which is $\max_{i\in S} l_i$. Then, coalition $N \setminus S$ picks last, which corresponds to its optimistic scenario. Knowing that a runway of length $\max_{i\in S} l_i$ has been built, it extends it, if needed, to a length of $\max_{i\in N} l_i$, which under the assumption that the valuations are large enough, leads to an efficient outcome. Hence, the optimistic and pessimistic approaches are dual for airport problems.

We conclude this section by an illustration of the line between duality and non-duality. In a cooperative production problem a set of agents share a production technology to produce some good(s). This joint production technology might exhibit increasing or decreasing returns to scale/scope.

If we suppose that demands for the good(s) are exogenous (see for instance Moulin and Shenker, 1992; Moulin, 1996; de Frutos, 1998), then producing these inelastic demands is efficient, and this is what we obtain by having S and $N \setminus S$ sequentially choose. Thus, the optimistic and pessimistic games are dual.

However, in a problem in which the efficient quantities demanded are endogenously determined (see, for instance Moulin, 1990; Roemer and Silvestre, 1993; Fleurbaey and Maniquet, 1996), then duality is lost, as S and $N \setminus S$ choosing sequentially can lead to under/over production and/or misallocation of the units produced. Thus, in such models, the optimistic and pessimistic approaches yield different results.

If we have decreasing returns to scale, an optimistic approach supposes that agents in S first decide how many units to consume, having access to the production technology for its first (low) marginal costs, which might induce them to overconsume compared to the optimal allocation. Our Theorem 2 guarantees that the anti-core of the optimistic game and the core of the pessimistic games are both non-empty. The pipeline externalities model of Trudeau and Rosenthal (2023), previously mentioned, also generalizes this case, if we suppose that damages occur only on the first node.

With increasing returns to scale, there is no such guarantee on the non-vacuity of the (anti-)core.

7. Concluding remarks

In the presence of externalities, defining a coalitional value game requires making assumptions on the behavior of other agents. For many problems, the use of an optimistic and a pessimistic approach leads to very natural ways to define such games, and these games form the extreme points of a large family of games. From a normative point of view, optimistic and pessimistic approaches should be approached with opposite solution concepts, as the former gives us upper bounds, while the latter provides lower bounds. In particular, if we use the core for games obtained from the pessimistic approach, we should use the anti-core for the optimistic approach.

We have shown that there is a great benefit in carefully defining these optimistic and pessimistic approaches. When the sign of the externalities are clearly defined, the optimistic/pessimistic approaches will clearly correspond to a coalition picking first or last, depending on the sign of these externalities. To properly define optimistic/pessimistic games that are complementary to each other, we need that the coalition choosing last supposes that the other agents have taken the optimal actions when choosing first. If we do so, we obtain the powerful result that the anti-core of the optimistic game is a subset of the core of the pessimistic game. If these externalities are negative, then we are guaranteed that the anti-core of the optimistic game is non-empty, and thus so is the core of the pessimistic game.

Moreover, we see that when the games defined based on whether a coalition chooses first or last are dual, it is redundant to define both the optimistic and pessimistic games. Hence, duality establishes the coincidence between the anti-core and the core and is easy to spot, and these greedy sequential choices still lead to efficient outcomes.

Our main results are useful in multiple ways. First, it clearly indicates which bounds are easier to satisfy, and even if we believe that the core of the pessimistic game is more interesting, the anti-core of the optimistic game is a refinement as we see that ensuring that nobody surpasses the optimistic bounds is more demanding than ensuring that nobody falls below the pessimistic bounds. In some applications, like in minimum cost spanning trees, this subset of allocations were shown to possess many interesting additional properties. Second, this also allows us an additional way to show that the core of the pessimistic game is non-empty. In fact, if externalities are negative, it is by going through the anti-core of the optimistic game that we can show that the core of the pessimistic game is always non-empty. Finally, when many optimistic/pessimistic variants have been proposed, for instance in the river sharing problem, our approach allows to better compare these games, as for each optimistic game we can define a corresponding pessimistic game, and vice versa.

Finally, in the presence of direct externalities, it is not entirely clear how the coalition choosing first should take into account the externalities imposed on others. We have considered the two extremes: it either ignores them entirely, or considers them as important as the impact on coalition members. Our main results can be generalized to cases where the coalition choosing first considers the impact on others, but not on the same level as the impacts on its members. Although this generalization is straightforward, we are unaware of applications that treat externalities in this manner. A possible open question is to see whether the anti-core of the optimistic and hence the core of the pessimistic game has a non-empty core under negative externalities when a coalition maximizes the welfare of all agents as in the pipeline externalities problems. The technique to prove Theorem 2, which focuses on the properties of the feasibility sets and revenue functions under negative externalities, does not extend to $\hat{v}^o$.

References

Ambec, S. and L. Ehlers (2008), "Sharing a river among satiable agents." *Games and Economic Behavior*, 64, 35–50.

Ambec, S. and Y. Sprumont (2002), "Sharing a river." *Journal of Economic Theory*, 107, 453–462.

Arribillaga, R. P. and G. Bergantiños (2022), "Cooperative and axiomatic approaches to the knapsack allocation problem." *Annals of Operations Research*, 318, 805–830.

Atay, A. and C. Trudeau (2024), "Queueing games with an endogenous number of machines." *Games and Economic Behavior*, 144, 104–125.

Aubin, J.-P. (1981), "Cooperative fuzzy games." *Mathematics of Operations Research*, 6, 1–13.

Béal, S., A. Ghintran, E. Rémila, and P. Solal (2012), "The river sharing problem: a survey." *International Game Theory Review*, 15, 1340016.

Bergantiños, G. and J. Vidal-Puga (2007a), "A fair rule in minimum cost spanning tree problems." *Journal of Economic Theory*, 137, 326–352.

Bergantiños, G. and J. Vidal-Puga (2007b), "The optimistic TU game in minimum cost spanning tree problems." *International Journal of Game Theory*, 36, 223–239.

Bergantiños, G. and J. Vidal-Puga (2015), "Characterization of monotonic rules in minimum cost spanning tree problems." *International Journal of Game Theory*, 44, 835–868.

Bird, C. G. (1976), "On cost allocation for a spanning tree: a game theoretic approach." *Networks*, 6, 335–350.

Bondareva, O. (1963), "Certain applications of the methods of linear programming to the theory of cooperative games." *Problemy Kibernetiki (in Russian)*, 10, 119–139.

Chun, Y. (2006), "A pessimistic approach to the queueing problem." *Mathematical Social Sciences*, 51, 171–181.

Chun, Y. (2016), *Fair queueing*. Springer.

Curiel, I. and S. Tijs (1991), "Minimarg and maximarg operators." *Journal of Optimization Theory and Applications*, 71, 277–287.

de Frutos, M. A. (1998), "Decreasing serial cost sharing under economies of scale." *Journal of Economic Theory*, 79, 245–275.

Feltkamp, V., S. Tijs, and S. Muto (1994), "On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems." Discussion Paper 1994-106, Tilburg University, Center for Economic Research.

Fleurbaey, M. and F. Maniquet (1996), "Cooperative production: A comparison of welfare bounds." *Games and Economic Behavior*, 17, 200–208.

Gillies, D. B. (1959), "Solutions to general non-zero-sum games." In *Contributions to the Theory of Games IV* (A. W. Tucker and R. D. Luce, eds.), 47–85, Princeton University Press.

Granot, D. and F. Granot (1992), "On some network flow games." *Mathematics of Operations Research*, 17, 792–841.

Hsiao, C. R. and T.E.S. Raghavan (1993), "Shapley value for multichoice cooperative games, I." *Games and Economic Behavior*, 5, 240–256.

Kalai, E. and E. Zemel (1982a), "Generalized network problems yielding totally balanced games." *Operations Research*, 30, 998–1008.

Kalai, E. and E. Zemel (1982b), "Totally balanced games and games of flow." *Mathematics of Operations Research*, 7, 476–478.

Kellerer, H., U. Pferschy, and D. Pisinger (2004), *Multidimensional knapsack problems*. Springer.

Khmelnitskaya, A. (2010), "Values for rooted-tree and sink-tree digraph games and sharing a river." *Theory and Decision*, 69, 657–669.

Kóczy, L. Á. (2018), *Partition function form games*, volume Theory and Decision Library C. Springer, Berlin, Germany.

Laurent-Lucchetti, J. and J. Leroux (2010), "Lindahl prices solve the NIMBY problem." *Economics Bulletin*, 30, 2457–2463.

Laurent-Lucchetti, J. and J. Leroux (2011), "Choosing and sharing." *Games and Economic Behavior*, 73, 296–300.

Littlechild, S. C. and G. Owen (1973), "A simple expression for the Shapley value in a special case." *Management Science*, 20, 370–372.

Maniquet, F. (2003), "A characterization of the shapley value in queueing problems." *Journal of Economic Theory*, 109, 90–103.

Moulin, H. (1990), "Uniform externalities: Two axioms for fair allocation." *Journal of Public Economics*, 43, 305–326.

Moulin, H. (1996), "Cost sharing under increasing returns: a comparison of simple mechanisms." *Games and Economic Behavior*, 13, 225–251.

Moulin, H. and S. Shenker (1992), "Serial cost sharing." *Econometrica*, 60, 1009–1037.

Oishi, T., M. Nakayama, H. Toru, and Y. Funaki (2016), "Duality and anti-duality in TU games applied to solutions, axioms, and axiomatizations." *Journal of Mathematical Economics*, 63, 44–53.

O'Neill, B. (1982), "A problem of rights arbitration from the talmud." *Mathematical Social Sciences*, 2, 345–371.

Peleg, B. and P. Sudhölter (2007), *Introduction to the theory of cooperative games*, 2nd edition. Springer.

Roemer, J. and J. Silvestre (1993), "The proportional solution for economies with both private and public ownership." *Journal of Economic Theory*, 59, 426–444.

Shapley, L. S. (1953), "A value for n-person games." In *Contributions to the Theory of Games II* (Kuhn. H. W. and A. W. Tucker, eds.), 307–317, Princeton University Press.

Shapley, L. S. (1967), "On balanced sets and cores." *Naval Research Logistics Quarterly*, 14, 453–460.

Shapley, L. S. (1971), "Cores of convex games." *International Journal of Game Theory*, 1, 11–26.

Tijs, S., S. Moretti, R. Branzei, and H. Norde (2006), "The Bird core for minimum cost spanning tree problems revisited: Monotonicity and additivity aspects." In *Recent Advances in Optimization*, 305–322, Springer.

Trudeau, C. and E. C. Rosenthal (2023), "The pipeline externalities game." Mimeo.

Van Essen, M. and J. Wooders (2023), "Dual auctions for assigning winners and compensating losers." *Economic Theory*, 76, 1069–1114.