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On bargaining sets of supplier-firm-buyer games

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H I G H L I G H T S

- We study a special three-sided matching game, the so-called supplier-firm-buyer game.
- We show that on this class the core and the Davis–Maschler bargaining set coincide.
- Moreover, the core also coincides with the Mas-Colell bargaining set on these games.
- Our results rest on the closedness of this class for taking the maximal excess game.

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A B S T R A C T

We study a special three-sided matching game, the so-called supplier-firm-buyer game, in which buyers and sellers (suppliers) trade indirectly through middlemen (firms). Stuart (1997) showed that all supplier-firm-buyer games have non-empty core. We show that for these games the core coincides with the classical bargaining set (Davis and Maschler, 1967), and also with the Mas-Colell bargaining set (Mas-Colell, 1989).

1. Introduction

In their seminal paper Shapley and Shubik (1972) introduced assignment games to study two-sided matching markets where there are indivisible goods which are traded between sellers and buyers in exchange for money. Their proof of the non-emptiness of the core established a fruitful research area. Multi-sided assignment games, however, have different features. Most importantly, the non-emptiness of the core is not guaranteed anymore even when there are only three sides in the game, as first demonstrated by Kaneko and Wooders (1982).

Since the core may be empty for multi-sided assignment games, some authors study conditions to obtain the non-emptiness of the core (see for instance Quint, 1991; Stuart, 1997; Sherstyuk, 1999; Atay and Núñez, 2017). In this paper, we focus on the class introduced by Brandenburger and Stuart (1996) and investigated by Stuart (1997). In the so-called supplier-firm-buyer games, agents in the market are partitioned into three sides and the groups are arranged along a chain. Sellers (suppliers) and buyers (customers) are at the two ends of the chain, but trade between them can only be made via agents in the middle (firms). The valuation on the supplier-firm-buyer triplets is locally additive, it sums up the potential values of the supplier-firm and of the firm-buyer matchings, but it is realized only if all three parties cooperate. Stuart (1997) showed that all supplier-firm-buyer games have non-empty core.

In order to find plausible payoff allocations even in games with empty core, Aumann and Maschler (1964) suggested a set-valued solution concept that incorporates some negotiating possibilities of the players. Among the various bargaining sets proposed, the one investigated by Davis and Maschler (1967) has emerged, for it was proved to be non-empty whenever the game has a non-empty imputation set (Davis and Maschler, 1967). Mas-Colell (1989) introduced another bargaining set notion based on preimputations

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and showed that it is non-empty for any game. (Holzman, 2001) proved that for superadditive games the classical (Davis–Maschler) bargaining set is included in the Mas-Colell bargaining set.

Solymosi (1999) presented a necessary and sufficient condition in terms of the so-called maximal excess games for the coincidence of the classical bargaining set and the core in superadditive games. Applied for two-sided assignment games, Solymosi (1999) proved the coincidence of the classical bargaining set and the core, by using the result of Granot and Granot (1992) who showed that the class of two-sided assignment games is closed for taking the maximal excess game at any imputation. Solymosi (2008) extended this closedness result to all preimputations in classes of partitioning games defined on a fixed family of basic coalitions and, by using the characterization by Holzman (2001) of the coincidence between the Mas-Colell bargaining set and the core, established even this stronger equivalence result for certain subclasses of partitioning games, including the two-sided assignment games.

In this paper, following a similar approach, we show that the class of supplier-firm-buyer games is closed for taking (the 0-normalization of) the maximal excess game at any (pre)imputation. Then, we establish the coincidence between the classical bargaining set and the core, and moreover the coincidence between the Mas-Colell bargaining set and the core for supplier-firm-buyer games. We restrict ourselves to the supplier-firm-buyer case, but all the arguments and results in the paper can be extended to m-sided assignment games with locally additive evaluation on the basic path-coalitions consisting exactly one agent from each side of the market. In real life, we observe markets that consist of more than two sides where the sectors are organized in a line. In such markets, agents from the same industry have an industry specific role and hence we cannot study these markets as separated two-sided markets. Thus, we believe that the generalization of supplier-firm-buyer games, namely multi-sided assignment games with locally additive value functions, is a useful model to study value generation and allocation in supply chains.

2. Preliminaries

A *transferable utility cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *0-normalized* if $v(\{i\}) = 0$ for every $i \in N$. It is *superadditive* if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$ for every two coalitions $S, T \subseteq N$.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i\in S} x_i$ if $S \neq \emptyset$ and $x(\emptyset) = 0$. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$; *individually rational*, if $x_i = x(\{i\}) \geq v(\{i\})$ for all $i \in N$; *coalitionally rational*, if $x(S) \geq v(S)$ for all $S \subseteq N$. The set of *preimputations*, $I^*(v)$, consists of the efficient payoff vectors, the set of *imputations*, $I(v)$, consists of the individually rational preimputations, and the *core*, $C(v)$, is the set of coalitionally rational (pre)imputations. We call a game *balanced* if its core is non-empty.

Given a game (N, v), the *excess* of a coalition $S \subseteq N$ at a payoff allocation x is $e_x(S) = v(S) - x(S)$. It is a measure of gain (or loss) to S, if its members disagree on x and leave it to form their own coalition. On player set N, games v and w are *strategically equivalent*, if there exist $\alpha > 0$ and $b \in \mathbb{R}^N$ such that $w(S) = \alpha v(S) + \sum_{i\in S} b_i$ for all $S \subseteq N$. In particular, the 0-normalization of v, denoted by v^0, is obtained when $\alpha = 1$ and $b = (b_i = -v(\{i\}) : i \in N)$. Clearly, v is balanced if and only if v^0 is balanced.

Aumann and Maschler (1964) argued that the purpose of the game is to reach some kind of stability, to which the players would or should agree, if they want any agreement. This stability should reflect in some sense the power of each player, but should be weaker than the sometimes too strong stability the core outcomes capture. Aumann and Maschler (1964) considered several bargaining sets as reasonable outcomes of negotiations among coalitions versus coalitions. Davis and Maschler (1967) investigated another variant, denoted $\mathbf{M}_1^i$, where individuals bargain with individuals and proved its non-emptiness under the very mild condition that the game has imputations. Hence, it received most attention and became the classical bargaining set. The idea behind is that an allocation can be considered stable (even if not in the core) if all objections raised by some player can be nullified by another player.

Let (N, v) be a coalitional game, $x \in I(v)$ be an imputation, and $i, j \in N$ be two different players. A pair (S, y) where $S \subseteq N$ and $y \in \mathbb{R}^S$ is an *objection of i against j at x* if $i \in S, j \notin S, y(S) = v(S)$, and $y_l > x_l$ for all $l \in S$. Then, a *counter-objection of j to the objection* (S, y) *of i at x* is a pair (T, z) such that $T \subseteq N$ and $z \in \mathbb{R}^T$ where $j \in T, i \notin T, z(T) = v(T), z_k \geq y_k$ for all $k \in T \cap S$, and $z_l \geq x_l$ for all $l \in T \setminus S$. An objection is *justified* (in the Davis–Maschler sense) if there does not exist any counter-objection to it. With these notions of objection and counter-objection, Davis and Maschler (1967) introduced what is known as the *classical bargaining set* $\mathbf{M}_1^i$.

Definition 1 (*Davis and Maschler, 1967*)**.** Let (N, v) be a coalitional game. The *classical bargaining set* is the set of imputations at which there is no justified objection:

$$\mathbf{M}_1^i(v) = \{x \in I(v) \mid \text{ for every objection at } x \text{ there is a counter-objection}\}.$$

Since no objections, hence no justified objections can be raised at core imputations, the core is always a subset of the classical bargaining set. Maschler (1976) discussed a five-player market game for which the bargaining set is a strict superset of the core, moreover, "for which the bargaining set seems to be intuitively more acceptable than the (non-empty) core". On the other hand, Solymosi (2002) proved that in (at most) 4-player games, if the core is non-empty, it coincides with the classical bargaining set.

Another bargaining set notion was introduced by Mas-Colell (1989). In that concept coalitions bargain, rather than pairs of players. Moreover, all efficient payoff vectors are considered, the individual rationality requirement is dropped. Thus, the notions of objection and counter-objection are modified.

Let (N, v) be a coalitional game. Given a preimputation $x \in I^*(N, v)$, we say that a pair (S, y) where $\emptyset \neq S \subseteq N$ and $y \in \mathbb{R}^S$ is a *weak objection* if $y(S) = v(S)$ and $y_l \geq x_l$ for all $l \in S$ with at least one strict inequality for some $l \in S$. Then, a pair (T, z) where $\emptyset \neq T \subseteq N$ and $z \in \mathbb{R}^T$ is a *strong counter-objection* to objection (S, y) at x if $z(T) = v(T)$ and $z_l \geq y_l$ for all $l \in T \cap S$, $z_l \geq x_l$ for all $l \in T \setminus S$ with at least one strict inequality for some $l \in T$. Using these concepts of weak objection and strong counter-objection, Mas-Colell (1989) introduced a notion of bargaining set.

Definition 2 (*Mas-Colell, 1989*)**.** Let (N, v) be a coalitional game. The *Mas-Colell bargaining set* is the set of preimputations such that every weak objection at the given preimputation can be strongly countered:

$$\mathbf{M}^*_{MC} = \{x \in I^*(v) | \text{every weak objection at } x \text{ can be strongly countered}\}.$$

Mas-Colell (1989) showed that the Mas-Colell bargaining set is non-empty in any game, and a superset of the core. Mas-Colell (1989) presented a 4-player market game where the Mas-Colell bargaining set contains imputations outside the (non-empty) core. On the other hand, it is easily seen that in (at most) 3-player games, if the core is non-empty, it coincides with the Mas-Colell bargaining set. Holzman (2001) showed that, for the superadditive games, the classical (Davis–Maschler) bargaining set is included in the Mas-Colell bargaining set, despite the seemingly not comparable notions of justified objection used in these two types of bargaining sets.

Given a game (N, v) and a fixed allocation $x \in \mathbb{R}^N$, the excess values define another game, called the *excess game at x*, on the same set of players N by the coalitional function $e_x(S) = v(S) - x(S)$ for all $S \subseteq N$. In a similar fashion, the *maximal excess game at x* is defined by the coalitional function $\widehat{e}_x(S) = \max_{T \subseteq S} e_x(T)$ for all $S \subseteq N$ on the same set of players N. Notice that the excess games are strategically equivalent to the game, but the maximal excess games are typically not. Observe that for each $x \in \mathbb{R}^N$, the maximal excess game $\widehat{e}_x$ is the *monotonic cover* of the excess game e_x, i.e. it is the minimal monotonic game such that $\widehat{e}_x(S) \geq e_x(S)$ for all $S \subseteq N$. Moreover, $\widehat{e}_x$ is non-negative; it is 0-normalized if x is individually rational (in particular when $x \in I(v)$); it is the constant null game if x is coalitionally rational (in particular when $x \in C(v)$). Finally, $\widehat{e}_x$ is superadditive if the game v is superadditive.

Making use of these induced games, Solymosi (1999) proved that in a superadditive game, if the maximal excess game $\widehat{e}_x$ at an imputation x outside the core is balanced, then there is a (Davis–Maschler type) justified objection, thus x cannot belong to the classical bargaining set. Holzman (2001) proved that the same condition is not just sufficient, but also necessary for the existence of a (Mas-Colell type) justified objection.

Theorem 1 (*Holzman, 2001*). *Given a coalitional game (N, v), let $x \in I^*(v) \setminus C(v)$. Then, $x \notin \mathbf{M}^*_{MC}(v)$ if and only if the maximal excess game $\widehat{e}_x$ has a non-empty core.*

Next, we show that for superadditive 0-normalized games, the 0-normalization of the maximal excess game at a preimputation is the same as the maximal excess game taken at the positive part of the preimputation.

Proposition 2. *Let (N, v) be a 0-normalized superadditive game. Then, for any preimputation x, the 0-normalization of the maximal excess game at x equals the maximal excess game taken at the payoff vector x^+ consisting of the positive parts of the payoffs, that is,*

$$\widehat{e}^{\,0}_x(S) = \widehat{e}_{x^+}(S) \quad \text{for all } S \subseteq N,$$

where $x^+_j = \max\{x_j, 0\}$ for all $j \in N$.

Proof. Let v be 0-normalized and superadditive. Clearly, since v is superadditive, both e_x and $\widehat{e}_x$ are also superadditive at all allocations $x \in \mathbb{R}^N$.

In case x is an imputation, the maximal excess game is 0-normalized. Furthermore, $x^+ = x$. Thus, $\widehat{e}^{\,0}_x = \widehat{e}_x = \widehat{e}_{x^+}$. So, our claim trivially holds.

Let $x \in I^*(v) \setminus I(v)$ and $N^-_x = \{j \in N : x_j < 0\}$. Clearly, $N^-_x \neq \emptyset$, $N^-_x \neq N$, and $\widehat{e}_x(j) = e_x(j) > 0$ for all $j \in N^-_x$.

First, we show that $\widehat{e}^{\,0}_x(S) \leq \widehat{e}_{x^+}(S)$ for all $S \subseteq N$. By superadditivity of $\widehat{e}_x$, if $\widehat{e}_x(S) = e_x(R)$ for some $R \subseteq S$, then $R \supseteq S \cap N^-_x$, and hence $R \cap N^-_x = S \cap N^-_x$. Then,

$$\begin{aligned}
\widehat{e}^{\,0}_x(S) &= \widehat{e}_x(S) - \sum_{k \in S \cap N^-_x} \widehat{e}_x(k) \\
&= e_x(R) - \sum_{k \in R \cap N^-_x} e_x(k) \\
&= v(R) - x(R) - \sum_{k \in R \cap N^-_x} (-x_k) \\
&= v(R) - x(R) - x^-(R) \qquad\qquad \text{with } x^- = x^+ - x \\
&= v(R) - x^+(R) = e_{x^+}(R) \\
&\leq \widehat{e}_{x^+}(R) \leq \widehat{e}_{x^+}(S)
\end{aligned}$$

where the inequalities follow from the definition and the monotonicity of the maximal excess game $\widehat{e}_{x^+}$.

Next, we will show the reverse inequalities $\widehat{e}^{\,0}_x(S) \geq \widehat{e}_{x^+}(S)$ hold for all $S \subseteq N$. First, notice that x^+ satisfies individual rationality in the 0-normalized game v. Let $\widehat{e}_{x^+}(S) = e_{x^+}(Q)$ for some $Q \subseteq S$, and assume that Q is the largest for inclusion among such coalitions. By superadditivity of $\widehat{e}_{x^+}$, we have $Q \supseteq S \cap N^-_x$, thus $Q \cap N^-_x = S \cap N^-_x$ and $x^-(Q) = x^-(S)$. Then,

$$\begin{aligned}
\widehat{e}_{x^+}(S) = e_{x^+}(Q) &= v(Q) - x^+(Q) \\
&= v(Q) - x(Q) - x^-(Q) \\
&= e_x(Q) - x^-(S) \\
&\leq \widehat{e}_x(Q) - \sum_{k \in S \cap N^-_x} e_x(k) \\
&\leq \widehat{e}_x(S) - \sum_{k \in S} \widehat{e}_x(k) = \widehat{e}^{\,0}_x(S)
\end{aligned}$$

where the inequalities follow from the definition and the monotonicity of the maximal excess game $\widehat{e}_x$. □

3. The supplier-firm-buyer market and game

We consider a market where there are three types of agents: suppliers, firms, and buyers. Each supplier has one unit of good to sell and each buyer would like to buy at most one unit of good where the trade between suppliers and buyers are made through firms.

Let N be the finite set of agents (players) in a market. They are partitioned in three sets N_1, N_2, and N_3 each of them representing one side of the market. A *three-sided assignment market* consists of three sides N_1, N_2, N_3, and a three-dimensional valuation matrix $A = (a_{ijk})_{i \in N_1, j \in N_2, k \in N_3}$ that represents the joint value that could be obtained by a triplet formed by exactly one agent from each side. Notice that negative joint values are allowed: if a buyer's willingness to pay is lower than a supplier's opportunity cost (see Brandenburger and Stuart, 1996 for further discussion on business value). Stuart (1997) studied the so-called supplier-firm-buyer market, where the value of a triplet formed by a supplier, a firm, and a buyer is generated by the separate partnerships between the supplier and the firm, and between the firm and the buyer. On the other hand, any of these pairwise partnerships is worthless unless completed by an agent of the third type.[1]

Definition 3 (*Stuart, 1997*). A three-sided assignment market $(N_1, N_2, N_3; A)$ is a *supplier-firm-buyer market* if there exist two matrices $B^1 = (b^1_{ij})_{(i,j) \in N_1 \times N_2}$ and $B^2 = (b^2_{jk})_{(j,k) \in N_2 \times N_3}$ such that $a_{ijk} = b^1_{ij} + b^2_{jk}$ for all $(i, j, k) \in N_1 \times N_2 \times N_3$.

Next, we introduce a cooperative game related to the supplier-firm-buyer market. The set of players is $N = N_1 \cup N_2 \cup N_3$. Since the smallest potentially valuable coalitions are the triplets formed by exactly one agent of each side, we define the triplets together with the single-player coalitions (representing the non-cooperating agents) as *basic coalitions*. We denote by

$$\mathcal{B} = \{\{i, j, k\} \mid (i, j, k) \in N_1 \times N_2 \times N_3\} \cup \{\{l\} \mid l \in N\}$$

the family of basic coalitions.

The worth of basic coalitions are defined as follows. Since in supplier-firm-buyer markets the positive added value of a triplet is assumed to be generated from the separate and independent trade between the supplier and the firm, and from the trade between the

[1] For a more general model, where the worth of a coalition is the sum of the amounts attached to all its pairs that belong to connected sides, see Atay and Núñez (2017).

firm and the buyer, the value of a triplet is obtained by summing its pairs' potential contributions, provided it is positive; otherwise, non-cooperation is more efficient:

$$w_A(\{i,j,k\}) = \max\{b^1_{ij} + b^2_{jk}, 0\} \tag{1}$$

for all $(i,j,k) \in N_1 \times N_2 \times N_3$. On the other hand, if an agent $l \in N$ does not participate in any trade, then her value is equal to zero, $w_A(\{l\}) = 0$.

The worth of non-basic coalitions are defined by the value of the most efficient partition of the coalitions into basic coalitions. Formally, a *basic partition* of a coalition is a family of pairwise disjoint basic coalitions whose union is the given coalition. Let $\mathcal{BP}(S)$ denote the set of basic partitions of $S \subseteq N$. Notice that if $S \neq \emptyset$ then $\mathcal{BP}(S) \neq \emptyset$. Then, the corresponding *supplier-firm-buyer game* is the pair (N, w_A) where $N = N_1 \cup N_2 \cup N_3$ is the set of players, and w_A is the coalitional function defined by $w_A(\emptyset) = 0$ and

$$w_A(S) = \max_{\mathcal{T} \in \mathcal{BP}(S)} \sum_{T \in \mathcal{T}} w_A(T) \tag{2}$$

for all $\emptyset \neq S \subseteq N$. The supplier-firm-buyer game is a special type of partitioning game introduced by Kaneko and Wooders (1982). It straightforwardly follows that these games are 0-normalized, non-negative, superadditive and monotonic.

Besides being a plausible model to study markets with middlemen, the supplier-firm-buyer game has important properties. Most notably, Stuart (1997) showed that it always has a non-empty core.

Proposition 3 (*Stuart, 1997*)**.** *Let $(N_1, N_2, N_3; A)$ be a supplier-firm-buyer market. Then, the corresponding supplier-firm-buyer game (N, w_A) has a non-empty core.*

We will show that for supplier-firm-buyer games, the core coincides with the Mas-Colell bargaining set, consequently, also with the classical bargaining set. To this end, first we show that the maximal excess game of a supplier-firm-buyer game taken at any preimputation is balanced, because it is either a supplier-firm-buyer game itself (for imputations) or the 0-normalization of a supplier-firm-buyer game (for preimputations which are not imputations).

Proposition 4. *Let (N, w_A) be a supplier-firm-buyer game. For any preimputation $(x,y,z) \in I^*(w_A)$, the maximal excess game $(N, \widehat{e}_{(x,y,z)})$ has a non-empty core.*

Proof. Let $\Gamma = (N_1, N_2, N_3; A)$ be a supplier-firm-buyer market and (N, w_A) be the corresponding game. Given a preimputation $(x,y,z) \in I^*(w_A)$, we show that the maximal excess game $(N, \widehat{e}_{(x,y,z)})$ is strategically equivalent to a supplier-firm-buyer game, hence it has a non-empty core.

CASE 1: $(x,y,z) \in C(w_A)$. Then, $\widehat{e}_{(x,y,z)}$ is the null game. That being so, it is the supplier-firm-buyer game corresponding to the market $(N_1, N_2, N_3; \overline{A} = \mathbf{0})$ with null evaluation of all basic coalitions. The null payoff vector is obviously in the core.

CASE 2: $(x,y,z) \notin C(w_A)$, but $(x,y,z) \geq (0,0,0)$, that is, (x,y,z) is individually rational in w_A, though efficiency is not assumed.

We construct an induced market $\overline{\Gamma} = (N_1, N_2, N_3; \overline{A})$ such that the corresponding game $\overline{w}_{\overline{A}}$ equals the maximal excess game $\widehat{e}_{(x,y,z)}$ of the initial game w_A. Consider the supplier-firm-buyer market $\overline{\Gamma}$ where the potential contribution of pair $(i,j) \in N_1 \times N_2$ to a trade is $\overline{b}^1_{ij} = b^1_{ij} - x_i - y^1_j$ and of a pair $(j,k) \in N_2 \times N_3$ is $\overline{b}^2_{jk} = b^2_{jk} - y^2_j - z_k$ with arbitrarily fixed non-negative payoffs $y^1_j \geq 0$ and $y^2_j \geq 0$ such that $y^1_j + y^2_j = y_j$ for all $j \in N_2$. It will turn out that the particular way we split the firms' (nonnegative) payoffs has no relevance. From the two matrices $\overline{B}^1 = (\overline{b}^1_{ij})_{(i,j)\in N_1 \times N_2}$ and $\overline{B}^2 = (\overline{b}^2_{jk})_{(j,k)\in N_2 \times N_3}$ we get the valuation of the triplets in the locally additive way: $\overline{a}_{ijk} = \overline{b}^1_{ij} + \overline{b}^2_{jk}$ for all $(i,j,k) \in N_1 \times N_2 \times N_3$. Then, the corresponding supplier-firm-buyer game $(N, \overline{w}_{\overline{A}})$ is obtained via (1) and (2).

First we show that $\widehat{e}_{(x,y,z)}(S) \leq \overline{w}_{\overline{A}}(S)$ for all $S \subseteq N$. It trivially holds, if $\widehat{e}_{(x,y,z)}(S) = 0$. Take an arbitrary coalition $S \subseteq N$ with $\widehat{e}_{(x,y,z)}(S) > 0$, and suppose that its maximal excess is achieved at coalition $Q \subseteq S$. Then, $e_{(x,y,z)}(Q) = \widehat{e}_{(x,y,z)}(S) \geq \widehat{e}_{(x,y,z)}(Q) \geq e_{(x,y,z)}(Q)$, so all inequalities hold as equalities. We may assume without loss of generality that Q is minimal for inclusion among such coalitions by dropping the single-player coalitions (whose excess and payoff are zero) from the optimal basic partition for Q. Then Q is a union of triplets, say $(i_1, j_1, k_1), \ldots, (i_m, j_m, k_m)$, all with positive excess at (x,y,z) in the initial game w_A. It follows that $w_A(\{i_l, j_l, k_l\}) = a_{i_l j_l k_l} > x_{i_l} + y_{j_l} + z_{k_l} \geq 0$. Then,

$$\begin{aligned}
\widehat{e}_{(x,y,z)}(S) = e_{(x,y,z)}(Q) &= \sum_{l=1}^{m} e_{(x,y,z)}(\{i_l, j_l, k_l\}) \\
&= \sum_{l=1}^{m} [(b^1_{i_l j_l} + b^2_{j_l k_l}) - (x_{i_l} + y_{j_l} + z_{k_l})] \\
&= \sum_{l=1}^{m} \left[(b^1_{i_l j_l} - x_{i_l} - y^1_{j_l}) + (b^2_{j_l k_l} - y^2_{j_l} - z_{k_l})\right] \\
&= \sum_{l=1}^{m} [\overline{b}^1_{i_l j_l} + \overline{b}^2_{j_l k_l}] \leq \sum_{l=1}^{m} \overline{w}_{\overline{A}}(\{i_l, j_l, k_l\}) \\
&\leq \overline{w}_{\overline{A}}(Q) \leq \overline{w}_{\overline{A}}(S),
\end{aligned}$$

where $y^1_{j_l} \geq 0$, $y^2_{j_l} \geq 0$ are the fixed payoff-splits such that $y^1_{j_l} + y^2_{j_l} = y_{j_l}$ for $l = 1, \ldots, m$. The first two inequalities come from the definition of the game $\overline{w}_{\overline{A}}$, the third inequality holds by the monotonicity of supplier-firm-buyer games.

Secondly, we show the reverse inequalities $\widehat{e}_{(x,y,z)}(S) \geq \overline{w}_{\overline{A}}(S)$ for all $S \subseteq N$. Again, it trivially holds, if $\overline{w}_{\overline{A}}(S) = 0$. Take an arbitrary coalition $S \subseteq N$ with $\overline{w}_{\overline{A}}(S) > 0$. Let $\{i_1, j_1, k_1\}, \ldots, \{i_m, j_m, k_m\}$ augmented with single-player coalitions $\{i_{m+1}\}, \ldots, \{i_p\}$, $\{j_{m+1}\}, \ldots, \{j_q\}$, $\{k_{m+1}\}, \ldots, \{k_r\}$ be an optimal basic partition of S in basic coalitions in the game $\overline{w}_{\overline{A}}$. We may assume that all triplets $\{i_l, j_l, k_l\}, (l = 1, \ldots, m)$, have a positive worth, $\overline{w}_{\overline{A}}(\{i_l, j_l, k_l\}) > 0$, $(l = 1, \ldots, m)$, since otherwise we can decompose them into singletons. Let $R \subseteq S$ be the union of the triplet coalitions in $\{i_l, j_l, k_l\}, (l = 1, \ldots, m)$. Then,

$$\begin{aligned}
\overline{w}_{\overline{A}}(S) &= \sum_{l=1}^{m} \overline{w}_{\overline{A}}(\{i_l, j_l, k_l\}) + \sum_{t \in S \setminus R} \overline{w}_{\overline{A}}(\{t\}) \\
&= \sum_{l=1}^{m} [\overline{b}^1_{i_l j_l} + \overline{b}^2_{j_l k_l}] \\
&= \sum_{l=1}^{m} \left[(b^1_{i_l j_l} - x_{i_l} - y^1_{j_l}) + (b^2_{j_l k_l} - y^2_{j_l} - z_{k_l})\right] \\
&= \sum_{l=1}^{m} \left[b^1_{i_l j_l} + b^2_{j_l k_l} - x_{i_l} - y_{j_l} - z_{k_l}\right] = \sum_{l=1}^{m} e_{(x,y,z)}(\{i_l, j_l, k_l\}) \\
&= e_{(x,y,z)}(R) \leq \widehat{e}_{(x,y,z)}(R) \leq \widehat{e}_{(x,y,z)}(S),
\end{aligned}$$

where the inequalities follow from the definition of the maximal excess game $\widehat{e}_{(x,y,z)}$.

Since imputations are individually rational, we get that the maximal excess game $\widehat{e}_{(x,y,z)}$ of supplier-firm-buyer game w_A taken at any $(x,y,z) \in I(w_A) \setminus C(w_A)$ is balanced, because it is the same as the supplier-firm-buyer game $\overline{w}_{\overline{A}}$ corresponding to the induced market $\overline{\Gamma}$.

CASE 3: $(x,y,z) \in I^*(w_A) \setminus I(w_A)$, that is, preimputation (x,y,z) violates individual rationality in the 0-normalized and superadditive game w_A. By Proposition 2, the 0-normalization of the maximal excess game $\widehat{e}^0_{(x,y,z)}$ at (x, y, z) equals the maximal excess game $\widehat{e}_{(x,y,z)^+}$ taken at the payoff vector $(x, y, z)^+$ consisting of the positive parts of the payoffs. Although payoff vector $(x, y, z)^+$ violates efficiency, it is individually rational in the game w_A. It follows from CASE 2 that the maximal excess game $\widehat{e}_{(x,y,z)^+}$ has a non-empty core. Being strategically equivalent to its 0-normalization, the maximal excess game $\widehat{e}_{(x,y,z)}$ has a non-empty core. □

Now, we can give our main result on the relationship between the core, the classical bargaining set, and the Mas-Colell bargaining set for supplier-firm-buyer games.

Theorem 5. *Let (N, w_A) be a supplier-firm-buyer game. Then, its core $C(w_A)$ coincides with the classical bargaining set $\mathbf{M}^i_1(w_A)$, and the Mas-Colell bargaining set $\mathbf{M}^*_{MC}(w_A)$,*

$$C(w_A) = \mathbf{M}^i_1(w_A) = \mathbf{M}^*_{MC}(w_A).$$

Proof. Since the supplier-firm-buyer game w_A is superadditive, by the comparability result of Holzman (2001), $C(w_A) \subseteq \mathbf{M}^i_1(w_A) \subseteq \mathbf{M}^*_{MC}(w_A)$. By Proposition 4, the maximal excess game at any preimputation has a non-empty core. Thus, by Theorem 1, no preimputation that is not in the core can belong to the Mas-Colell bargaining set. Consequently, all three set-valued solutions coincide. □

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References

Atay, A., Núñez, M., 2017. Multi-sided assignment games on m-partite graphs, UB Economics Working Papers E17/357.
Aumann, R.J., Maschler, M., 1964. The bargaining set for cooperative games. In: Advances in Game Theory. Princeton University Press, Princeton, pp. 443–476.
Brandenburger, A., Stuart, H., 1996. Value-based business strategy. J. Econ. Manag. Strategy 5, 5–24.
Davis, M., Maschler, M., 1967. Existence of stable payoff configurations for cooperative games. In: Shubik, M. (Ed.), Essays in Mathematical Economics in Honour of Oskar Morgenstern. Princeton University Press, Princeton, pp. 39–52.
Granot, D., Granot, F., 1992. On some network flow games. Math. Oper. Res. 17, 792–841.
Holzman, R., 2001. The comparability of the classical and the Mas-Colell bargaining sets. Internat. J. Game Theory 29, 543–553.
Kaneko, M., Wooders, M., 1982. Cores of partitioning games. Math. Social Sci. 3, 313–327.
Mas-Colell, A., 1989. An equivalence theorem for a bargaining set. J. Math. Econom. 18, 129–139.
Maschler, M., 1976. An advantage of the bargaining set over the core. J. Econom. Theory 13, 184–192.
Quint, T., 1991. The core of an m-sided assignment game. Games Econom. Behav. 3, 487–503.
Shapley, L., Shubik, M., 1972. The assignment game I: The core. Internat. J. Game Theory 1, 111–130.
Sherstyuk, K., 1999. Multisided matching games with complementarities. Internat. J. Game Theory 28, 489–509.
Solymosi, T., 1999. On the bargaining set, kernel and core of superadditive games. Internat. J. Game Theory 28, 229–240.
Solymosi, T., 2002. The bargaining set of four-person balanced games. Internat. J. Game Theory 31, 1–11.
Solymosi, T., 2008. Bargaining sets and the core in partitioning games. CEJOR Cent. Eur. J. Oper. Res. 16, 425–440.
Stuart, H., 1997. The supplier-firm-buyer game and its m-sided generalization. Math. Social Sci. 34, 21–27.