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A note on the relationship between the core and stable sets in three-sided markets

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HIGHLIGHTS

- The relationship on the core and stable sets in three-sided markets is studied.
- Dominant diagonal is a necessary condition for the core to be a stable set.
- Besides, it is also a sufficient condition when each side of the market has two agents.
- We extend to three-sided assignment games the notion of the μ-compatible subgame.
- We show the union of the cores of μ-compatible subgames may not be a stable set.

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ABSTRACT

We analyze the extent to which two known results of the relationship between the core and the stable sets for two-sided assignment games can be extended to three-sided assignment games. We find that the dominant diagonal property is necessary for the core to be a stable set and, likewise, sufficient when each sector of the three-sided market has two agents. Unlike the two-sided case, the union of the extended cores of all the μ-compatible subgames with respect to an optimal matching μ may not be a von Neumann–Morgenstern stable set.

1. Introduction

In this paper, we consider markets with three different sectors or sides. Coalitions of agents can only achieve a non-negative joint profit by means of triplets comprising one agent of each side. Then, a three-dimensional valuation matrix represents the joint profit of all these possible triplets. These markets, introduced by Kaneko and Wooders (1982), are a generalization of the two-sided assignment games first introduced by Shapley and Shubik (1972). In a similar vein, Stuart (1997) represents a supplier-firm-buyer situation using a three-sided assignment market.

In a two-sided assignment game, the two sectors are associated with a sector of buyers and a sector formed by sellers. Each seller has one unit of an indivisible good to sell and each buyer wants to buy at most one unit. The valuation matrix represents the joint profit obtained by each buyer–seller transaction. From these valuations a coalitional game is obtained and the total profit under an optimal matching between buyers and sellers yields the worth of the grand coalition. A distribution of this worth such that each agent receives at least his/her individual coalition worth is called an *imputation*. The best known solution concept for the coalitional game is the *core*. Roughly speaking, a dominance relation is defined between imputations and the core is the set of undominated imputations.

Three-sided assignment markets appear naturally when a supplier (or middleman) is needed to match a buyer with a seller, or when each buyer needs to buy two complementary objects from two different types of seller to make a profit (for instance, each buyer needs to buy a computer as well as the services of an internet provider). A key difference between the two-sided and the three-sided assignment games is that while the core is always non-empty in the case of the former, it may be empty in that of the latter (Kaneko and Wooders, 1982). This is why we are interested in the studying of some other set-valued solution concepts, such as stable sets, for these games.

A *von Neumann–Morgenstern stable set* (von Neumann and Morgenstern, 1944) is a set of imputations that satisfies internal stability and external stability: *(a)* no imputation in the set is dominated by any other imputation in the set and *(b)* each imputation outside the set is dominated by some imputation in the set. It is known from Lucas (1968) that a game may have no stable set. Since the core always satisfies internal stability, it is included in any stable set; and if the core is externally stable, then it is the only stable set.

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Other notions of stability are analyzed in Roth (1976), Peris and Subiza (2013), and Han and van Deemen (2016).

The purpose of this paper is to analyze the extent to which existing results for two-sided markets can be extended to three-sided markets. In particular, we focus on two existing results of the relationship between the core and the stable sets. First, we study whether the dominant diagonal property is a necessary and sufficient condition for the core to be a stable set, as in two-sided markets (Solymosi and Raghavan, 2001). Second, we analyze whether the union of the extended cores of all μ-compatible subgames is a stable set of the three-sided assignment game, as in the two-sided markets (Núñez and Rafels, 2013).

In the case of two-sided assignment games, Solymosi and Raghavan (2001) show that the core of a two-sided assignment game is a stable set if and only if the valuation matrix has a dominant diagonal. Later, Núñez and Rafels (2013) prove the existence of a stable set for all two-sided assignment games. The stable set they introduce is the only one to exclude third-party payments with respect to an optimal matching μ and it is defined through certain subgames, known as μ-compatible subgames.

In the present paper, we generalize the notion of the dominant diagonal to the three-sided case and prove that it is a necessary condition for the core of this game to be a stable set. We also show that for three-sided markets with only two agents on each side, the dominant diagonal property suffices to guarantee that the core is stable. It remains open as to whether it is also sufficient for arbitrary three-sided assignment markets. Furthermore, we extend the notion of μ-compatible subgames introduced by Núñez and Rafels (2013) to the three-sided case. Then, given a three-sided game and an optimal matching μ, we consider the set V^{μ} formed by the union of the cores of all μ-compatible subgames. In contrast with the two-sided case, we show that V^{μ} may not be a stable set. However, we prove that V^{μ} is an abstract core if we restrict the set of feasible payoff vectors to those imputations that are compatible with μ. Moreover, V^{μ} coincides with the usual core if and only if the valuation matrix has a dominant diagonal.

The rest of the paper is organized as follows. In Section 2 we outline the preliminaries on assignment games. Section 3 extends the notion of a dominant diagonal valuation matrix and studies its relationship with core stability. Finally, in Section 4, we extend the notion of μ-compatible subgames, and show that the union of their cores may not be a stable set but that it still satisfies some appealing property.

2. Preliminaries

Let $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$ be pairwise disjoint countable sets. An $m \times m \times m$ *assignment market* $\gamma = (M_1, M_2, M_3; A)$ consists of three different sectors with m agents each: $M_1 = \{1, 2, \ldots, m\} \subseteq \mathcal{U}_1$, $M_2 = \{1', 2', \ldots, m'\} \subseteq \mathcal{U}_2$, $M_3 = \{1'', 2'', \ldots, m''\} \subseteq \mathcal{U}_3$, and a three-dimensional valuation matrix $A = \langle (a_{ijk}) : i \in M_1, j \in M_2, k \in M_3 \rangle$ that represents the potential joint profit obtained by triplets comprising one agent from each side. These triplets are the *basic coalitions* of the three-sided assignment game, as defined by Quint (1991).

Given subsets of agents of each sector, $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, and $S_3 \subseteq M_3$, a *matching* μ for the submarket $\gamma_{|S} = (S_1, S_2, S_3; A_{|S_1 \times S_2 \times S_3})$ is a subset of the Cartesian product, $\mu \subseteq S_1 \times S_2 \times S_3$, such that each agent belongs to at most one triplet. We denote by $\mathcal{M}(S_1, S_2, S_3)$ the set of all possible matchings. A matching $\mu \in \mathcal{M}(S_1, S_2, S_3)$ is an *optimal matching* for the submarket if

$$\sum_{(i,j,k)\in\mu} a_{ijk} \geq \sum_{(i,j,k)\in\mu'} a_{ijk}$$

for all $\mu' \in \mathcal{M}(S_1, S_2, S_3)$. We denote by $\mathcal{M}_A(S_1, S_2, S_3)$ the set of all optimal matchings for the submarket $(S_1, S_2, S_3; A_{|S_1 \times S_2 \times S_3})$.

The $m \times m \times m$ *assignment game*, (N, w_A), related to the above assignment market has a player set $N = M_1 \cup M_2 \cup M_3$ and a characteristic function

$$w_A(S) = \max_{\mu \in \mathcal{M}(S\cap M_1, S\cap M_2, S\cap M_3)} \sum_{(i,j,k)\in\mu} a_{ijk}$$

for all $S \subseteq N$. In the sequel, we need to exclude some agents. Then, if we exclude some agents $I \subseteq M_1$, $J \subseteq M_2$, and $K \subseteq M_3$, we can write $w_{A_{-I\cup J\cup K}}$ instead of $w_{A_{|(M_1\setminus I)\times(M_2\setminus J)\times(M_3\setminus K)}}$. Note that these subgames need not have the same number of agents in each sector. Nevertheless, the notion of matching and characteristic function is defined analogously as for the $m \times m \times m$ case.

Given an $m \times m \times m$ assignment game, a payoff vector, or an allocation, is $(u, v, w) \in \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+$ where u_l denotes the payoff[1] to agent $l \in M_1$, v_l denotes the payoff to agent $l' \in M_2$ and w_l denotes the payoff to agent $l'' \in M_3$. An *imputation* is a non-negative payoff vector that is efficient, $u(M_1) + v(M_2) + w(M_3) = \sum_{i\in M_1} u_i + \sum_{j\in M_2} v_j + \sum_{k\in M_3} w_k = w_A(M_1 \cup M_2 \cup M_3)$. We denote the set of imputations of the assignment game (N, w_A) by $I(w_A)$.

Given an optimal matching $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$, we define the *$\mu$-principal section* of (N, w_A) as the set of payoff vectors such that $u_i + v_j + w_k = a_{ijk}$ for all $(i, j, k) \in \mu$ and the payoff to agents unassigned by μ is zero. We denote this by $B^{\mu}(w_A)$. Note that $B^{\mu}(w_A) \subseteq I(w_A)$. In the μ-principal section, the only side payments that take place are those between agents matched together by μ.

We can assume that the optimal matching is on the main diagonal of the valuation matrix, $\mu = \{(i, i', i'') | i \in \{1, 2, \ldots, m\}\}$. Notice that the allocation $(a, 0, 0)$, that is $u_i = a_{iii}$ for all $i \in M_1$, $v_j = w_k = 0$ for all $j \in M_2$, $k \in M_3$, always belongs to the μ-principal section. The same is true of the allocations $(0, a, 0)$ and $(0, 0, a)$. These three vertices of the polytope $B^{\mu}(w_A)$ are known as the *sector-optimal allocations*. The core of a game is the set of imputations (u, v, w) such that no coalition S can improve upon: $u(S \cap M_1) + v(S \cap M_2) + w(S \cap M_3) \geq w_A(S)$. In our case, it is easy to see that it is enough to consider individual and basic coalitions. An imputation (u, v, w) belongs to the core, $(u, v, w) \in C(w_A)$, if and only if for all $(i, j, k) \in M_1 \times M_2 \times M_3$ it holds $u_i + v_j + w_k \geq a_{ijk}$. Note that, together with efficiency, the above constraints imply that the core is a subset of the μ-principal section for any optimal matching μ.

It is well-known (see Kaneko and Wooders, 1982) that the core of a three-sided assignment game may be empty. For the particular case in which each sector contains only two agents, Lucas (1995) gives necessary and sufficient conditions for balancedness (that is non-emptiness of the core). Under the assumption that $\mu = \{(1, 1', 1''), (2, 2', 2'')\}$ is an optimal matching, the core of a $2 \times 2 \times 2$ assignment game is non-empty if and only if it satisfies the following conditions:

$$\begin{aligned} &2a_{111} + a_{222} \geq a_{112} + a_{121} + a_{211}, \\ &a_{111} + 2a_{222} \geq a_{221} + a_{212} + a_{122}. \end{aligned} \tag{1}$$

Given a three-sided assignment market $\gamma = (M_1, M_2, M_3; A)$, we define a binary relation on the set of imputations, namely the *dominance relation*. Given two imputations (u, v, w) and (u', v', w'), we say (u, v, w) *dominates* (u', v', w') if and only if there exists $(i, j, k) \in M_1 \times M_2 \times M_3$ such that $u_i > u'_i$, $v_j > v'_j$, $w_k > w'_k$ and $u_i + v_j + w_k \leq a_{ijk}$. We denote it by $(u, v, w) \operatorname{dom}^A_{\{i,j,k\}} (u', v', w')$. We write $(u, v, w) \operatorname{dom}^A (u', v', w')$ to denote that (u, v, w) dominates (u', v', w') by means of some triplet (i, j, k).[2] Given a set of

[1] $\mathbb{R}^m_+$ is the m-dimensional real vector space with non-negative coordinates. Hence, to simplify notation, we only consider individually rational payoff vectors.

[2] This dominance relation is the usual one introduced by von Neumann and Morgenstern (1944). It is clear that in the case of multi-sided assignment games, we only need to consider domination via basic coalitions. When no confusion regarding the valuation matrix can arise, we will simply write $(u, v, w) \operatorname{dom} (u', v', w')$.

imputations $V \subseteq I(w_A)$, we denote by $D(V)$ the set of imputations dominated by some element in V and by $U(V)$ those imputations not dominated by any element in V.

Two main set-valued solution concepts are defined by means of this dominance relation: the core and the stable set. The core, whenever it is non-empty, coincides with the set of undominated imputations; that is, $C(w_A) = U(I(w_A))$. The other solution concept defined by means of domination is the von Neumann–Morgenstern stable set.

A subset of the set of imputations, $V \subseteq I(w_A)$, is a *von Neumann–Morgenstern solution* or a *stable set* if it satisfies internal and external stability:

(i) *internal stability:* for all $(u, v, w), (u', v', w') \in V$, (u, v, w) $\mathrm{dom}^A(u', v', w')$ does not hold,
(ii) *external stability:* for all $(u', v', w') \in I(w_A) \setminus V$, there exists $(u, v, w) \in V$ such that $(u, v, w)\ \mathrm{dom}^A(u', v', w')$.

Internal stability of a set of imputations V guarantees that no imputation of V is dominated by another imputation of $V : V \subseteq U(V)$. The core is internally stable. External stability imposes that all imputations outside V are dominated by an imputation in V : $I(w_A) \setminus V \subseteq D(V)$. In general, the core fails to satisfy external stability. Both conditions (internal and external stability) can be summarized as $V = U(V)$.

The next section addresses the first aim of this paper, namely, the study of the stability of the core of a three-sided assignment game.

3. Core stability

In this section, we study whether the known core stability results for two-sided assignment games can be extended to the three-sided case. We begin by generalizing the dominant diagonal property, introduced by Solymosi and Raghavan (2001) for two-sided assignment games, to the three-sided case. They prove that this condition characterizes the core stability of the two-sided case. We assume that the valuation matrix is square, that is, that each side has the same number of agents on each side. Note that, whenever necessary, we can assume without loss of generality that an optimal matching is placed on the main diagonal.

Definition 1. Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$. Matrix A has a *dominant diagonal* if and only if for all $i \in \{1, 2, \ldots, m\}$ it holds

$$a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\} \quad \text{for all } j, k \in \{1, 2, \ldots, m\}.$$

Clearly, if A has a dominant diagonal, then $\mu = \{(i, i', i'') \mid i \in \{1, 2, \ldots, m\}\}$ is an optimal matching.

As in the two-sided case, the dominant diagonal property characterizes those markets in which giving the profit of each optimal partnership to one given sector leads to a core element.

Proposition 1. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$. The valuation matrix A has a* dominant diagonal *if and only if all sector-optimal allocations belong to the core.*

Proof. First, we prove the "*if*" part. Take the sector-optimal allocation for the first sector: $(u, v, w) = (a_{111}, \ldots, a_{mmm}; 0, \ldots, 0; 0, \ldots, 0)$. If it belongs to the core, then we have $a_{iii} = u_i = u_i + v_j + w_k \geq a_{ijk}$ for all $(i, j, k) \in M_1 \times M_2 \times M_3$. For the remaining allocations, the proof is analogous.

To prove the "*only if*" part, let A be a three-dimensional valuation matrix with the dominant diagonal property. By Definition 1, for all $i \in \{1, 2, \ldots, m\}$ and for all $j, k \in \{1, 2, \ldots, m\}$, $a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\}$. If we take the sector-optimal allocation $(u, v, w) = (a_{111}, \ldots, a_{mmm}; 0, \ldots, 0; 0, \ldots, 0)$, the above inequality trivially shows that it belongs to the core. Analogously, $(0, a, 0)$ and $(0, 0, a)$ are also core allocations. □

The above proposition provides a characterization of the dominant diagonal property. The fact that the sector-optimal core allocations are in the core does not depend on the selected optimal matching means that the dominant diagonal property is independent of the optimal matching placed on the main diagonal.

The following proposition shows that the dominant diagonal property is necessary for the stability of the core.

Proposition 2. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and an optimal matching on the main diagonal. If the core is a von Neumann–Morgenstern stable set, then its corresponding valuation matrix A has a dominant diagonal.*

Proof. Let us suppose, on the contrary, that the core of a three-sided assignment game (N, w_A) is a von Neumann–Morgenstern stable set but its corresponding three-dimensional valuation matrix A does not have a dominant diagonal. If A does not have a dominant diagonal, then, by Proposition 1, there exists one sector-optimal allocation, let us say $(a, 0, 0)$, that does not belong to the core. But then, since $C(w_A)$ is assumed to be a von Neumann–Morgenstern stable set, there exists $(u', v', w') \in C(w_A)$ such that

$$(u', v', w')\ \mathrm{dom}_{\{i,j,k\}}(a, 0, 0).$$

Then, $u'_i > u_i = a_{iii}$ which contradicts $(u', v', w') \in C(w_A)$. □

Proposition 2 raises the question of the equivalence between the von Neumann–Morgenstern stability of the core and the dominant diagonal property of the matrix. That is, if A has a dominant diagonal, is the core of the assignment game, $C(w_A)$, a von Neumann–Morgenstern stable set? We can answer this question in the affirmative when the market has only two agents in each sector. The proof is consigned to the working paper (Atay and Núñez, 2018).

Proposition 3. *Given a $2 \times 2 \times 2$ assignment game (N, w_A) with an optimal matching on the main diagonal, the core $C(w_A)$ is a von Neumann–Morgenstern stable set if and only if A has a dominant diagonal.*

In the section that follows, we return to the general case, that is to say, $m \times m \times m$ assignment games, in search of a stable set.

4. The union of the cores of all μ-compatible subgames

We follow an approach similar to that employed in Núñez and Rafels (2013) when constructing a stable set for two-sided assignment markets. To this end, we first extend the notion of the μ-compatible subgame to three-sided assignment games. Then, we consider the union of the extended cores of all μ-compatible subgames and we identify the stability properties of this set. We show that it may not satisfy external stability, and hence, unlike the two-sided case, it does not always result in a von Neumann–Morgenstern stable set.

Definition 2. Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game, with $m = |M_1| = |M_2| = |M_3|$, $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$ an optimal matching, and $I \subseteq M_1$, $J \subseteq M_2$ and $K \subseteq M_3$. The subgame

$$(M_1 \setminus I, M_2 \setminus J, M_3 \setminus K, w_{A_{-I \cup J \cup K}})$$

is a *μ-compatible subgame* if and only if

$$w_A(M_1 \cup M_2 \cup M_3) = w_A((M_1 \setminus I) \cup (M_2 \setminus J) \cup (M_3 \setminus K)) + \sum_{\substack{(i,j,k)\in\mu \\ i\in I}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ j\in J}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ k\in K}} a_{ijk}.$$

When a subgame is μ-compatible, each agent outside the subgame can leave the market taking with them the full profit of their partnership in the optimal matching μ, and what remains is exactly the worth of the submarket. As a consequence, any core element of the subgame can be completed with the payoffs of the excluded agents in order to obtain an imputation of the initial market. By so doing, the (extended) cores of the μ-compatible subgames can be seen as set-valued solutions for the original game, either to complement the core with more possible allocations that may result from cooperation or to replace the core whenever it is empty.

Without loss of generality, assume that the diagonal matching is an optimal matching for A: $\mu = \{(i, i', i'') | i \in \{1, 2, \dots, m\}\}$. Then, given a μ-compatible subgame $w_{A_{-I\cup J\cup K}}$ we define its extended core,

$$\hat{C}(w_{A_{-I\cup J\cup K}}) = \left\{ (x, z) \in B^{\mu}(w_A) \;\middle|\; \begin{array}{l} x_i = a_{iii} \text{ for all } i \in I \cup J \cup K, \\ z \in C(w_{A_{-I\cup J\cup K}}) \end{array} \right\}.$$

In particular, if we exclude all agents in M_1, then the game $(N \setminus M_1, w_{A_{-M_1}})$ is always a μ-compatible subgame since $w_{A_{-M_1}}(N \setminus M_1) = 0$. The core of this μ-compatible subgame is reduced to $\{(0, 0)\} \subseteq \mathbb{R}^{M_2} \times \mathbb{R}^{M_3}$ and its extended core is $\hat{C}(w_{A_{-M_1}}) = \{(a, 0, 0)\}$. Analogous μ-compatible subgames are obtained when we exclude the agents of one of the remaining sides of the market.

Given a three-sided assignment market $\gamma = (M_1, M_2, M_3; A)$, we define the set of all coalitions that give rise to μ-compatible subgames:

$$\mathcal{C}^{\mu}(A) = \{R \subseteq M_1 \cup M_2 \cup M_3 \mid w_{A_{-R}} \text{ is a } \mu\text{-compatible subgame}\}.$$

Note that when $R = \emptyset$ we retrieve the core of the initial game (N, w_A).

Now, for any assignment market $\gamma = (M_1, M_2, M_3; A)$, we define the set $V^{\mu}(w_A)$ formed by the union of extended cores of all μ-compatible subgames:

$$V^{\mu}(w_A) = \bigcup_{R \in \mathcal{C}^{\mu}(A)} \hat{C}(w_{A_{-R}}) \tag{2}$$

An immediate consequence of the above definition is that $V^{\mu}(w_A)$ is a subset of the μ-principal section:

$$V^{\mu}(w_A) \subseteq B^{\mu}(w_A).$$

Note also that, unlike the core, the set $V^{\mu}(w_A)$ is always non-empty since it contains at least the three points $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, a)$, that result from the μ-compatible subgames in which all the agents of one sector have been excluded. In fact, the following example shows that $V^{\mu}(w_A)$ can be reduced to just these three points and, hence, it can be non-convex and disconnected.

Example 1. Consider a three-sided assignment game in which each sector has two agents, $M_1 = \{1, 2\}$, $M_2 = \{1', 2'\}$, and $M_3 = \{1'', 2''\}$, and the valuation matrix A is the following

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{3} & 1 \\ 2 & 5 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 1 & 4 \\ 5 & \mathbf{4} \end{pmatrix} \\ 2'' \end{array} .$$

Notice there is a unique optimal matching $\mu = \{(1, 1', 1''), (2, 2', 2'')\}$. By Lucas' conditions for balancedness, see (1), we can see that the core is empty: $a_{111} + 2a_{222} = 11 < 14 = a_{221} + a_{122} + a_{212}$. We observe that the only μ-compatible subgames are $w_{A_{-\{1,2\}}}$, $w_{A_{-\{1',2'\}}}$ and $w_{A_{-\{1'',2''\}}}$. Hence $V^{\mu}(w_A) = \{(a, 0, 0), (0, a, 0), (0, 0, a)\} = \{(3, 4; 0, 0; 0, 0), (0, 0; 3, 4; 0, 0), (0, 0; 0, 0; 3, 4)\}$. Now it is easy to realize that such points do not dominate any imputation in the μ-principal section. Thus, $V^{\mu}(w_A)$ is not externally stable, which implies it is not a von Neumann–Morgenstern stable set.[3]

[3] V^{μ} may also not be a subsolution, as defined in Roth (1976). See Atay and Núñez (2018) for a detailed discussion of this same example.

Although V^{μ} is not a stable set, this set does have some appealing stability properties in three-sided markets. Recall the notion of the core of an abstract game in Lucas (1992), based on the definition of a set of feasible outcomes and a dominance relation in this set. Thus, our abstract core consists of the undominated allocations in the μ-principal section $B^{\mu}(w_A)$, rather than taking all the undominated imputations as in the usual definition of the core. Note that in the setting of three-sided markets, it makes sense to assume that the feasible outcomes are those in a μ-principal section, where utility is only transferred between those agents that are matched each other.

To this end, we state a property of the set $V^{\mu}(w_A)$ that allows us to check whether an allocation belongs to this set without there being any need to compute all the compatible subgames. We omit the proof since it is quite straightforward.

Lemma 4. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and an optimal matching μ on the main diagonal. Let (u, v, w) be an allocation of the principal section, that is, $(u, v, w) \in B^{\mu}(w_A)$. Then $(u, v, w) \in V^{\mu}(w_A)$ if and only if for all $(i, j, k) \in M_1 \times M_2 \times M_3$ at least one of the four following statements holds:*

(i) either $u_i = a_{iii}$
(ii) or $v_j = a_{jjj}$
(iii) or $w_k = a_{kkk}$
(iv) or $u_i + v_j + w_k \geq a_{ijk}$.

The following proposition shows that when we consider the outcomes in the μ-principal section as feasible, the set $V^{\mu}(w_A)$ is precisely the set of undominated outcomes.

Proposition 5. *Let $(M_1 \cup M_2 \cup M_3, w_A)$ be a three-sided assignment game with $|M_1| = |M_2| = |M_3| = m$, and $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$. Then,*

$$V^{\mu}(w_A) = U(B^{\mu}(w_A))$$

where $U(B^{\mu}(w_A))$ is the set of imputations that are undominated by the μ-principal section.

Proof. Let us write $V = V^{\mu}(w_A)$ and assume μ is on the main diagonal. First, we prove $U(B^{\mu}(w_A)) \subseteq B^{\mu}(w_A)$. Note that this inclusion is equivalent to $I(w_A) \setminus B^{\mu}(w_A) \subseteq D(B^{\mu}(w_A))$, where $D(B^{\mu}(w_A))$ is the set of imputations that are dominated by some allocation in the μ-principal section.

Take $(x, y, z) \in I(w_A) \setminus B^{\mu}(w_A)$. Then, there exists $i \in \{1, \dots, m\}$ such that $x_i + y_i + z_i < a_{iii}$. Take $\varepsilon = a_{iii} - x_i - y_i - z_i > 0$, and define λ_1, λ_2 and λ_3 by $\lambda_1 = \frac{x_i + \frac{\varepsilon}{3}}{a_{iii}}$, $\lambda_2 = \frac{y_i + \frac{\varepsilon}{3}}{a_{iii}}$ and $\lambda_3 = \frac{z_i + \frac{\varepsilon}{3}}{a_{iii}}$. Note that $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_1 a_{iii} = x_i + \frac{\varepsilon}{3} > x_i$, $\lambda_2 a_{iii} = y_i + \frac{\varepsilon}{3} > y_i$ and $\lambda_3 a_{iii} = z_i + \frac{\varepsilon}{3} > z_i$.

Now, recall that $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$ all belong to $B^{\mu}(w_A)$ and take the point $(u, v, w) = \lambda_1(a, 0, 0) + \lambda_2(0, a, 0) + \lambda_3(0, 0, a) \in B^{\mu}(w_A)$. Then, for all $i \in \{1, \dots, m\}$, $u_i + v_i + w_i = (\lambda_1 + \lambda_2 + \lambda_3)a_{iii} = a_{iii}$. Together with $u_i > x_i$, $v_i > y_i$ and $w_i > z_i$, this implies that $(u, v, w) \operatorname{dom}_{\{i,i',i''\}} (x, y, z)$ and hence $(x, y, z) \in D(B^{\mu}(w_A))$.

Now we prove $V = U(B^{\mu}(w_A))$. First, we show $V \subseteq U(B^{\mu}(w_A))$, that is, no allocation in V is dominated by an allocation in the μ-principal section. Consider two allocations $(u, v, w) \in B^{\mu}(w_A)$ and $(u', v', w') \in V$. Assume that for some $(i, j, k) \in M_1 \times M_2 \times M_3$, $(u, v, w) \operatorname{dom}_{\{i,j,k\}} (u', v', w')$ holds, which means $u_i + v_j + w_k \leq a_{ijk}$ together with $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$.

Note that $(u', v', w') \notin C(w_A)$, since core elements are undominated. Hence, $(u', v', w') \in \hat{C}(w_{A_{-R}})$ for some $R \in \mathcal{C}^{\mu}(A)$.

If $i \in R$, then $u'_i = a_{iii}$. Then $u_i > u'_i = a_{iii}$ which contradicts $(u, v, w) \in B^{\mu}(w_A)$. The same argument leads to a contradiction if $j \in R$ or $k \in R$. If $i \notin R, j \notin R$ and $k \notin R$, then by Lemma 4, $u'_i + v'_j + w'_k \geq a_{ijk} \geq u_i + v_j + w_k$ which contradicts our assumption $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$. This finishes the proof of $(u, v, w) \in U(B^\mu(w_A))$.

Now, we move to $U(B^\mu(w_A)) \subseteq V$. Assume on the contrary that $(u, v, w) \in U(B^\mu(w_A))$ and $(u, v, w) \notin V$. Since $U(B^\mu(w_A)) \subseteq B^\mu(w_A)$, $(u, v, w) \in B^\mu(w_A)$. Then, $(u, v, w) \in B^\mu(w_A)$ and $(u, v, w) \notin V$ which implies by Lemma 4 there exists $(i, j, k) \in M_1 \times M_2 \times M_3$ such that $u_i < a_{iii}$, $v_j < a_{jjj}$, $w_k < a_{kkk}$ and $u_i + v_j + w_k < a_{ijk}$. Define $\varepsilon_1 = a_{iii} - u_i > 0$, $\varepsilon_2 = a_{jjj} - v_j > 0$, $\varepsilon_3 = a_{kkk} - w_k > 0$ and $\varepsilon_4 = a_{ijk} - u_i - v_j - w_k > 0$. Also, let us define $u'_i = u_i + \min\{\varepsilon_1, \frac{\varepsilon_4}{3}\}$, $v'_j = v_j + \min\{\varepsilon_2, \frac{\varepsilon_4}{3}\}$ and $w'_k = w_k + \min\{\varepsilon_3, \frac{\varepsilon_4}{3}\}$. Note that $u'_1 > u_1$, $v'_j > v_j$, $w'_k > w_k$ and $u'_i + v'_j + w'_k < u_i + v_j + w_k + 3\frac{\varepsilon_4}{3} = a_{ijk}$. Now, we can complete the definition of $(u', v,' w')$ as follows:

Since, by definition, $u'_i \leq a_{iii}$, define $v'_i = a_{iii} - u'_i$ and $w'_i = 0$. Similarly, since $v'_j \leq a_{jjj}$, define $u'_j = a_{jjj} - v'_j$ and $w'_i = 0$. And finally, since $w'_k \leq a_{kkk}$, define $v'_k = a_{kkk} - w'_k$ and $u'_k = 0$. For all $l \in \{1, \dots, m\} \setminus \{i, j, k\}$ define $u'_l = a_{lll}$, $v'_l = 0$ and $w'_l = 0$. Then $(u', v', w') \in B^\mu(w_A)$ and (u', v', w') dom $_{\{i,j,k\}}(u, v, w)$ which contradicts $(u, v, w) \in U(B^\mu(w_A))$. □

The above proposition shows that for a given optimal matching μ, $V^\mu(w_A)$ is the "core", if we take as our set of feasible payoff vectors the μ-principal section $B^\mu(w_A)$. We have shown that $V^\mu(w_A)$ is always non-empty and, moreover, it can be easily proved that the set $V^\mu(w_A)$ coincides with the usual core $C(w_A)$ if and only if the valuation matrix A has a dominant diagonal.

In Proposition 5 we also show that there is no allocation in the μ-principal section that dominates any element of $V^\mu(w_A)$. This ensures the internal stability of $V^\mu(w_A)$. As a result, if for some given optimal matching μ there exists a stable set included in the μ-principal section, then $V^\mu(w_A)$ would be included in this stable set.

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