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An alternative proof of the characterization of core stability for the assignment game

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ABSTRACT

Solymosi and Raghavan (2001), characterize the stability of the core of the assignment game by means of a property of the valuation matrix. They show that the core of an assignment game is a von Neumann–Morgenstern stable set if and only if its valuation matrix has a dominant diagonal. While their proof makes use of graph-theoretical tools, the alternative proof presented here relies on the notion of the buyer–seller exact representative, as introduced by Núñez and Rafels in 2002.

1. Introduction and preliminaries

An *assignment market* $(M, M'; A)$ consists of two different sectors: let us say a finite set of buyers M and a finite set of sellers M' (M and M' disjoint) and a non-negative valuation matrix $A = (a_{ij})_{\substack{i\in M\\ j\in M'}}$ that represents the potential joint profit obtained by each mixed-pair $(i, j) \in M \times M'$. As in [4,2], we assume that the assignment market is *square*, that is $|M| = |M'|$.

A *matching* μ between M and M' is a subset of the Cartesian product, $M \times M'$, such that each agent belongs, at most, to one pair. The set of all possible matchings is denoted by $\mathcal{M}(M, M')$. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the market (M, M', A) if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}_A(M, M')$ the set of all optimal matchings for the market (M, M', A). The corresponding *assignment game* $(M \cup M', w_A)$ has a player set $M \cup M'$ and a characteristic function $w_A(S \cup T) = \max_{\mu\in\mathcal{M}(S,T)} \sum_{(i,j)\in\mu} a_{ij}$ for all $S \subseteq M$ and $T \subseteq M'$.

In this paper, we assume without loss of generality that $\mu = \{(i, i) \mid i \in M\}$, the matching refers to the main diagonal, is optimal. We use "j" to denote both the jth buyer and the jth seller, since the distinction is always clear from the context.

Once a matching between buyers and sellers that maximizes the total profit in the market has been chosen, we need to determine how this profit can be allocated among the agents. Given an assignment game $(M \cup M', w_A)$, an *allocation* is a payoff vector $(u; v) \in \mathbb{R}^{|M|} \times \mathbb{R}^{|M'|}$, where u_l denotes the payoff to buyer $l \in M$ and v_l denotes the payoff to seller $l \in M'$. An *imputation* is a payoff vector that is efficient, $\sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(M \cup M')$ and individually rational, $u_i \geq w_A(\{i\}) = 0$ for all $i \in M$ and $v_j \geq w_A(\{j\}) = 0$ for all $j \in M'$. We denote the set of imputations of an assignment game $(M \cup M', w_A)$ by $I(w_A)$.

In an assignment game, the *principal section* consists of imputations that satisfy $u_i + v_i = a_{ii}$ for all $i \in M$. We denote it by $B(w_A)$. In the principal section, side-payments only take place between matched agents. There are two special and useful allocations (named the *sector-optimal allocations*): $(\boldsymbol{a}; \mathbf{0}) \in \mathbb{R}^{|M|} \times \mathbb{R}^{|M'|}$, defined by $a_k = a_{kk}$ for $k \in M$ and $a_k = 0$ for $k \in M'$, and $(\mathbf{0}; \boldsymbol{a}) \in \mathbb{R}^{|M|} \times \mathbb{R}^{|M'|}$ defined by $a_k = 0$ for $k \in M$ and $a_k = a_{kk}$ for $k \in M'$.

A binary relation, known as *domination*, is defined on the set of imputations. Given two imputations $(u; v)$ and $(u'; v')$, we say that $(u; v)$*dominates*$(u'; v')$ if and only if there exists $(i, j) \in M \times M'$ such that $u_i > u'_i$, $v_j > v'_j$ and $u_i + v_j \leq a_{ij}$. We then write $(u; v)\ \mathrm{dom}^A_{\{i,j\}}(u'; v')$, and $(u; v)\ \mathrm{dom}^A(u'; v')$ to denote that $(u; v)$ dominates $(u'; v')$ by means of some pair $(i, j) \in M \times M'$. Such definition only makes use of mixed-pair allocations, and for assignment games it is equivalent to the usual dominance relation in [5].

The first solution concept for coalitional games that appears in the literature is the notion of stable set. A subset V of the set of imputations $I(w_A)$ is a *von Neumann–Morgenstern stable set* (a stable set) if it satisfies *internal stability*, that is, for any $(u; v), (u'; v') \in V$, $(u; v)\ \mathrm{dom}^A(u'; v')$ does not hold; and *external stability*, that is, for any $(u'; v') \in I(w_A) \setminus V$, there exists $(u; v) \in V$ such that $(u; v)\ \mathrm{dom}^A(u'; v')$.

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The *core* $C(w_A)$ is another solution concept that can also be defined, see [1], by means of the dominance relation: it is the set of undominated imputations.

Equivalently, an imputation $(u; v) \in I(w_A)$ belongs to the core of the assignment game $C(w_A)$ if for all $(i,j) \in M \times M'$ it holds $u_i + v_j \geq a_{ij}$. It is shown in [3] that an assignment game $(M \cup M', w_A)$ always has a non-empty core. Notice that the core always satisfies internal stability but may fail to satisfy external stability. This raises the question as to which valuation matrices correspond to assignment games with an externally stable (and hence stable) core.

Solymosi and Raghavan introduce in [4] the *dominant diagonal* property for valuation matrices. A square valuation matrix A has a dominant diagonal if all diagonal elements are row and column maxima: $a_{ii} \geq \max\{a_{ij}, a_{ji}\}$ for all $(i,j) \in M \times M'$. Hence, an optimal matching is placed on the main diagonal. It is straightforward to see that a valuation matrix A has a dominant diagonal if and only if the sector-optimal allocations $(\boldsymbol{a}; \mathbf{0})$ and $(\mathbf{0}; \boldsymbol{a})$ belong to the core. It is then proved in [4] that *"the core of a square assignment game $(M \cup M', w_A)$ is a von Neumann–Morgenstern stable set if and only if the valuation matrix A has a dominant diagonal"*. The authors' proof is based on graph-theoretical arguments while here we base ours on the properties of the buyer–seller exact representative of an assignment game proposed in [2].

Given any assignment game $(M \cup M', w_A)$, there exists a unique valuation matrix A^r such that $C(w_A) = C(w_{A^r})$ and A^r is the maximum with this property. That is, if any entry in A^r is raised, the resulting game has a different core. As a consequence, if the matrix A^r is the buyer–seller exact representative of A, then for all $(i,j) \in M \times M'$ there exists $(u, v) \in C(w_{A^r})$ such that $u_i + v_j = a^r_{ij}$. Notice that for each $(i,j) \in M \times M'$, a^r_{ij} is the lower bound for the joint payoff of agents $i \in M$ and $j \in M'$ in the core.

Based on [2], we are now able to offer a proof of the characterization of core stability for assignment games alternative to that provided in [4]. The advantage of this new approach is that it relies solely on the structure of the assignment game, that is, on the known bounds for the payoff to each mixed-pair in the core. For this reason, it might be possible to apply these ideas to the characterization of core stability for markets with more than two sectors, which, to the best of our knowledge, remains an open question.

2. Core stability

In this section, we provide the main result of this paper, an alternative proof of the characterization of core stability for the two-sided assignment game.

To do so, we first show that when the valuation matrix has a dominant diagonal, each core allocation can be connected to the two sector-optimal allocations, $(\boldsymbol{a}; \mathbf{0})$ and $(\mathbf{0}; \boldsymbol{a})$, by a continuous monotonic curve with parameter τ. Shapley claims this lemma but does not offer the proof.

Lemma 1. *Let $(M \cup M', w_A)$ be a square two-sided assignment game such that its valuation matrix A has a dominant diagonal. Given any vector belonging to the core of the game, $(u; v) \in C(w_A)$, and any $\tau \in \mathbb{R}$, the vector $(u(\tau); v(\tau))$ defined by*

$$u_i(\tau) = med\,\{0, u_i - \tau, a_{ii}\} \quad \text{for all } i \in M, \tag{1}$$

$$v_i(\tau) = med\,\{0, v_i + \tau, a_{ii}\} \quad \text{for all } i \in M',$$

belongs to $C(w_A)$.

Proof. Note first that for $\tau_1 = \max_{i\in M} a_{ii}$, $(u(\tau_1); v(\tau_1)) = (\mathbf{0}; \boldsymbol{a})$ and for $\tau_2 = -\tau_1$, $(u(\tau_2); v(\tau_2)) = (\boldsymbol{a}; \mathbf{0})$. Notice that since $(u; v) \in C(w_A)$, we have $u_i + v_i = a_{ii}$ for all $i \in M$ and hence, for all $\tau \in \mathbb{R}$ and all $i \in M$

$$v_i + \tau = a_{ii} - u_i + \tau = a_{ii} - (u_i - \tau). \tag{2}$$

It is then straightforward to show that $u_i(\tau) + v_i(\tau) = a_{ii}$ for all $i \in M$.

Take now $i \neq j$ and consider three different cases to check that $(u(\tau); v(\tau))$ satisfies the core constraints:

1. $\tau < -\min\{v_i, v_j\}$, that is either $u_i(\tau) = a_{ii}$ or $v_j(\tau) = 0$. In the first case, $u_i(\tau) = a_{ii}$, we have $u_i(\tau) + v_j(\tau) \geq u_i(\tau) = a_{ii} \geq a_{ij}$ where the last inequality follows from the dominant diagonal assumption. Otherwise, if $u_i(\tau) < a_{ii}$ and $v_j(\tau) = 0$, then since $v_j(\tau) = \text{med}\{0, v_j + \tau, a_{jj}\}$, $v_j + \tau \leq 0$. This implies $\tau \leq 0$ and also $u_i(\tau) + v_j(\tau) = u_i - \tau \geq u_i + v_j \geq a_{ij}$, where the last inequality follows from (u, v) being in the core.
2. $\tau > \min\{u_i, u_j\}$, that is either $v_j(\tau) = a_{jj}$ or $u_i(\tau) = 0$. If $v_j(\tau) = a_{jj}$ then $u_i(\tau) + v_j(\tau) \geq a_{jj} \geq a_{ij}$ because of the dominant diagonal assumption. If $v_j(\tau) < a_{jj}$ but $u_i(\tau) = 0$, since $u_i(\tau) = \text{med}\{0, u_i - \tau, a_{ii}\}$, we have $u_i - \tau \leq 0$. Then, $\tau \geq 0$ and hence $u_i(\tau) + v_j(\tau) = v_j(\tau) = v_j + \tau \geq v_j + u_i \geq a_{ij}$, where the last inequality follows from (u, v) being in the core.
3. $-\min\{v_i, v_j\} \leq \tau \leq \min\{u_i, u_j\}$. This implies $u_i(\tau) = u_i - \tau$ and $v_j(\tau) = v_j + \tau$ and hence, again from (u, v) being in the core, $u_i(\tau) + v_j(\tau) = u_i + v_j \geq a_{ij}$. □

Next, to show that the core of a square two-sided assignment game is a von Neumann–Morgenstern stable set if and only if its valuation matrix has a dominant diagonal, we need to prove the following lemma that states a property of the principal section.

Lemma 2. *Let $(M \cup M', w_A)$ be a square two-sided assignment game with an optimal matching on the main diagonal. Given $(x; y) \in B(w_A) \setminus C(w_A)$, there exist a pair $(i,j) \in M \times M'$ and a core allocation $(u; v) \in C(w_A)$ such that $x_i + y_j < a_{ij} = u_i + v_j$.*

Proof. From [2], for any assignment game $(M \cup M', w_A)$ there exists another assignment game $(M \cup M', w_{A^r})$ with the same core, $C(w_A) = C(w_{A^r})$, and A^r maximum with this property. Hence, if $(x; y) \notin C(w_A)$, then $(x; y) \notin C(w_{A^r})$. This means $x_i + y_j < a^r_{ij}$ for some $(i,j) \in M \times M'$ and there exists a core allocation $(u; v)$ such that

$$x_i + y_j < a^r_{ij} = u_i + v_j. \tag{3}$$

If $a^r_{ij} = a_{ij}$, the lemma is proved. Otherwise, by the definition of A^r, see page 433 in [2], $a^r_{ij} = a_{ik_1} + a_{k_1k_2} + a_{k_2k_3} + \cdots + a_{k_rj} - a_{k_1k_1} - \cdots - a_{k_rk_r}$ for some $k_1, \ldots, k_r \in M \setminus \{i,j\}$ and all of them different.

Since $(u; v)$ is a core allocation and the main diagonal is an optimal matching,

$$\begin{aligned} u_i + v_j &= a_{ik_1} + a_{k_1k_2} + \cdots + a_{k_rj} - a_{k_1k_1} - \cdots - a_{k_rk_r} \\ &= a_{ik_1} + a_{k_1k_2} + \cdots + a_{k_rj} \\ &\quad - (u_{k_1} + v_{k_1}) - \cdots - (u_{k_r} + v_{k_r}). \end{aligned} \tag{4}$$

By rearranging (4), we obtain

$$u_i + v_{k_1} + u_{k_1} + v_{k_2} + \cdots + u_{k_r} + v_j = a_{ik_1} + \cdots + a_{k_rj}. \tag{5}$$

From $(u; v) \in C(w_A)$, and (5), we obtain

$$u_{l_1} + v_{l_2} = a_{l_1l_2}$$

for all $(l_1, l_2) \in \{(i, k_1), (k_1, k_2), \ldots, (k_{r-1}, k_r), (k_r, j)\}$.

Since $(x; y) \in B(w_A)$, we know $x_t + y_t = a_{tt} = u_t + v_t$ for all $t \in \{k_1, k_2, \ldots, k_r\}$. Now,

$$\begin{aligned} x_i + y_{k_1} + x_{k_1} + y_{k_2} + \cdots + x_{k_r} + y_j &= x_i + y_j + \sum_{l=1}^{r} x_{k_l} + y_{k_l} \\ &< u_i + v_j + \sum_{l=1}^{r} u_{k_l} + v_{k_l} \\ &= u_i + v_{k_1} + u_{k_1} + v_{k_2} + \cdots + u_{k_r} + v_j \\ &= a_{ik_1} + \cdots + a_{k_rj}, \end{aligned}$$

where the inequality follows from (3) and the last equality follows from (5).

Then, $x_i+y_{k_1}+x_{k_1}+y_{k_2}+\cdots+x_{k_r}+y_j < a_{ik_1}+a_{k_1k_2}+\cdots+a_{k_rj}$. This means that $x_i + y_{k_1} < a_{ik_1} = u_i + v_{k_1}$, or $x_{k_r} + y_j < a_{k_rj} = u_{k_r} + v_j$, or $x_{k_l} + y_{k_{l+1}} < a_{k_lk_{l+1}} = u_{k_l} + v_{k_{l+1}}$ for some $l \in \{1, \ldots, r-1\}$. □

We can now state and prove the main result.

Theorem 3. *Let* $(M \cup M', w_A)$ *be a square assignment game with an optimal matching on the main diagonal. Then the following statements are equivalent:*

(i) *A has a dominant diagonal,*
(ii) $C(w_A)$ *is a von Neumann–Morgenstern stable set.*

Proof. We first consider (i) ⇒ (ii). The core of an assignment game is always internally stable, thus we only need to settle the external stability. When an allocation $(x; y)$ belongs to $I(w_A) \setminus B(w_A)$, Shapley shows in his personal notes that it is dominated by some core allocations. Here, we reproduce the proof for completeness. Assume $(x; y) \in I(w_A) \setminus B(w_A)$. Since $(x; y) \in I(w_A)$ and $\mu = \{(k, k)|k \in M\}$ is an optimal matching, $\sum_{k\in M} x_k + y_k = \sum_{k\in M} a_{kk}$. Moreover, since $(x; y) \notin B(w_A)$, there is some $i \in M$ such that $x_i + y_i \neq a_{ii}$. We can assume $x_i + y_i < a_{ii}$ since if $x_{i'} + y_{i'} > a_{i'i'}$ for some $i' \in M$, because of $\sum_{k\in M} x_k + y_k = \sum_{k\in M} a_{kk}$, there is $i \in M \setminus \{i'\}$ with $x_i+y_i < a_{ii}$. Thus, $x_i < a_{ii}-y_i$, which implies that there exists $0 \leq x_i < \lambda < a_{ii} - y_i \leq a_{ii}$. By Lemma 1, there exists $(u; v) \in C(w_A)$ with $u_i = \lambda$. Then, $u_i > x_i$ and $u_i < a_{ii} - y_i$ which implies $y_i < a_{ii} - u_i = v_i$. Moreover, $x_i + y_i < a_{ii} = u_i + v_i$. Hence, $(u; v)\ \mathrm{dom}^A_{\{i,i\}} (x; y)$.

Assume now that $(x; y) \in B(w_A) \setminus C(w_A)$. We know by Lemma 2 that there exist a pair $(i, j) \in M \times M'$ and $(u; v) \in C(w_A)$ such that $x_i + y_j < a_{ij} = u_i + v_j$. Now, assume without loss of generality $u_i > x_i$. If also $v_j > y_j$, we obtain $(u; v)\ \mathrm{dom}^A_{\{i,j\}} (x; y)$.

Otherwise, assume $v_j \leq y_j$. Since both $(x; y)$ and $(u; v)$ belong to $B(w_A)$, $x_j + y_j = u_j + v_j = a_{jj}$. Then, $u_j \geq x_j$. Notice that $u_i > x_i + y_j - v_j = x_i + (u_j + v_j - x_j) - v_j = x_i + u_j - x_j$. Hence,

$$u_j - u_i + x_i < x_j. \tag{6}$$

We want to show that a core allocation exists that dominates $(x; y)$ via coalition $\{i, j\}$. To this end, we consider some cases:

1. $x_j > 0$. Consider two cases:
   (a) $0 \leq x_i < a_{ii}$. Consider the continuous monotonic curve defined as in (1) through $(u; v)$, and take the point corresponding to $\tau^\varepsilon = u_i - x_i - \varepsilon$ where $0 < \varepsilon \leq a_{ii} - x_i$. We prove that for some $0 < \varepsilon \leq a_{ii} - x_i$, $(u(\tau^\varepsilon); v(\tau^\varepsilon))$ dominates $(x; y)$ via $\{i, j\}$. Notice that, for all $0 < \varepsilon \leq a_{ii} - x_i$, $u_i(\tau^\varepsilon) = \mathrm{med}\{0, u_i - u_i + x_i + \varepsilon, a_{ii}\} = x_i + \varepsilon > x_i$. Now, since $u_j(\tau^\varepsilon) = \mathrm{med}\{0, u_j - u_i + x_i + \varepsilon, a_{jj}\}$ and by (6) $u_j - u_i + x_i < x_j \leq a_{jj}$, there exists $0 < \varepsilon_1 \leq a_{ii} - x_i$ small enough such that $u_j(\tau^{\varepsilon_1}) \neq a_{jj}$. Then, we examine two cases:
      i. $u_j(\tau^{\varepsilon_1}) = u_j - u_i + x_i + \varepsilon_1$. Notice that $u_i(\tau^{\varepsilon_1}) > x_i$, $u_j(\tau^{\varepsilon_1}) < x_j$ or equivalently $v_j(\tau^{\varepsilon_1}) > y_j$ which together with $u_i(\tau^{\varepsilon_1}) + v_j(\tau^{\varepsilon_1}) = u_i + v_j = a_{ij}$ proves $(u(\tau^{\varepsilon_1}); v(\tau^{\varepsilon_1}))\ \mathrm{dom}^A_{\{i,j\}} (x; y)$.
      ii. $u_j(\tau^{\varepsilon_1}) = 0 < x_j$. Then, $v_j(\tau^{\varepsilon_1}) = a_{jj} > y_j$. Moreover, $v_j(\tau^{\varepsilon_1}) = a_{jj}$ implies $v_j(\tau^{\varepsilon_1}) \leq v_j + \tau^{\varepsilon_1}$. Since $u_i(\tau^{\varepsilon_1}) = x_i+\varepsilon = u_i-\tau^{\varepsilon_1}$, we have $u_i(\tau^{\varepsilon_1})+v_j(\tau^{\varepsilon_1}) \leq u_i+v_j = a_{ij}$. Together with $v_j(\tau^{\varepsilon_1}) > y_j$ and $u_i(\tau^{\varepsilon_1}) > x_i$ this implies that $(u(\tau^{\varepsilon_1}); v(\tau^{\varepsilon_1}))\ \mathrm{dom}^A_{\{i,j\}} (x; y)$.
   (b) $x_i = a_{ii}$. Since, by assumption, $u_i > x_i$, we obtain $a_{ii} = x_i < u_i$ which contradicts $(u; v) \in C(w_A)$.
2. $x_j = 0$. Since $(x; y) \in B(w_A)$, $y_j = a_{jj}$. We obtain from $x_i + y_j < a_{ij}$ that $a_{jj} < a_{ij}$, which contradicts the dominant diagonal assumption regarding the valuation matrix.

This shows that any $(x; y) \in B(w_A) \setminus C(w_A)$ is dominated by a core allocation via coalition $\{i, j\}$, which concludes the proof of (i) ⇒ (ii).

Next, we prove (ii) ⇒ (i). Suppose A does not have a dominant diagonal, then the sector-optimal allocation $(\boldsymbol{a}; \mathbf{0})$ does not belong to the core. On the premise that $C(w_A)$ is a von Neumann–Morgenstern stable set, there exists $(u; v) \in C(w_A)$ such that $(u; v)\ \mathrm{dom}^A_{\{i,j\}} (\boldsymbol{a}; \mathbf{0})$ for some $(i, j) \in M \times M'$. So that, $u_i > a_{ii}$, a contradiction of $(u; v) \in C(w_A)$. □

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