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Interfaces with Other Disciplines

Optimistic and pessimistic approaches for cooperative games ☆

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ABSTRACT

Cooperative game theory explores how to fairly allocate the joint value generated by a group of decision-makers, but its application is compromised by the large number of counterfactuals needed to compute the value of all coalitions, a problem made even more complicated when externalities are present. We provide a theoretical foundation for a simplification used in many applications, in which the value of a coalition is computed assuming that they either select before or after the complement set of agents, providing optimistic and pessimistic values on what a coalition should receive. In a vast set of problems exhibiting what we call feasibility externalities, we show that ensuring a coalition does not receive more than its optimistic value is always at least as difficult as ensuring it receives its pessimistic value. Furthermore, under the presence of negative externalities, we establish the existence of stable allocations that respect these bounds. Finally, we examine well-known optimization-based applications and their corresponding cooperative games to show how our results lead to new insights and allow the derivation of further results from the existing literature.

1. Introduction

Cooperative transferable utility (TU) games provide a powerful framework for analyzing collaboration among decision-makers, offering tools to allocate the value or cost of joint projects in a fair and stable way. This paper adds to the stream of research on operations research games emerging from scenarios where a coalition of players must solve a shared optimization problem.[1] These cooperative tools have been applied extensively across a broad range of operations research applications, including the distribution of revenues in streaming platforms (Schlicher et al., 2024; Bergantiños & Moreno-Ternero, 2025a; Gonçalves-Dosantos et al., 2025), cost-sharing in manufacturing (Atay et al., 2021; Alon & Anily, 2023; Munich, 2024), international kidney exchange programs (Benedek et al., 2025), resource allocation problem (Rahmoune et al., 2024), claims problems (O'Neill, 1982), airport cost allocation (Littlechild & Owen, 1973), and joint production problems (Moulin, 1990; Moulin & Shenker, 1992). However, the practical application of cooperative game theory becomes significantly more complex in the presence of externalities—when the actions of players outside a coalition influence what that coalition can achieve.

In particular, when assessing the value of a coalition $S \subseteq N$, the assumptions we make about the behavior of the complement $N \setminus S$ play a central role. Are the outsiders cooperating among themselves? Are they competing with or actively trying to harm S? A formal treatment of these possibilities would require using partition form games, which requires building a very large number of counterfactuals, compromising their practical use.

In many applications based on an optimization problem, such as queueing (Chun, 2016), minimum cost spanning trees (Bird, 1976), or river-sharing problems (Ambec & Sprumont, 2002), simpler assumptions have been adopted: namely, that the coalition either moves before or after the rest of the players. In this paper, we provide a general framework that unifies and justifies these modeling choices through the concept of feasibility externalities, where the feasible action set of a coalition depends on the actions taken by outsiders. Our model not

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[1] See Borm et al. (2001) for a survey and Curiel (2013) for an overview of this literature.

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only provides a general framework for existing models but also offers a foundation for analyzing new classes of operations research games arising from an underlying joint optimization problem.

We distinguish between negative and positive feasibility externalities. With negative externalities, the actions of non-members reduce the feasible set available to a coalition—so a worst-case value arises when the coalition moves last, and a best-case value arises when it moves first. Conversely, under positive externalities, where the actions of non-members expand a coalition's feasible set, the worst-case scenario corresponds to the coalition moving first, and the best-case one to it moving last. These two perspectives yield natural lower and upper bounds on what a coalition can achieve, forming the basis of what we call the pessimistic and optimistic value functions.

We then explore the core (Gillies, 1959) of the pessimistic game and the anti-core (see, for instance, Oishi et al., 2016) of the optimistic game, identifying allocations that respectively ensure coalitions receive at least their lower bound or no more than their upper bound. A key result is that the anti-core of the optimistic game is always a subset of the core of the pessimistic game. This inclusion provides a powerful simplification: ensuring that no coalition receives more than its optimistic value is always at least as hard as guaranteeing it receives its pessimistic value—and in the case of negative externalities, such stable allocations are always guaranteed to exist.

By examining applications like queueing and minimum cost spanning tree problems, we show that our approach allows to recover natural concepts studied in these particular problems. For instance, we obtain, from a different perspective, the irreducible core for minimum cost spanning tree problems. Additionally, we provide a condition to identify applications in which the pessimistic and optimistic functions form dual games. Examples include well-studied problems like bankruptcy (O'Neill, 1982) and airport cost allocation (Littlechild & Owen, 1973). The condition is linked to the optimization itself: if a coalition S choosing first and its complement $N \setminus S$ choosing last always yields an efficient outcome, then the two games are dual, and there is no gain in applying the optimistic and pessimistic approaches separately.

1.1. Related literature

We provide here a brief discussion of how our optimistic and pessimistic coalitional value functions compare to other concepts proposed in the literature.

Different methods have been proposed to determine the value of a coalition starting with the α and β games (Shubik, 1982; pp. 136–138). The alpha game corresponds to maximin (a coalition maximizes its revenues after the complement set has tried to minimize it) while the beta game corresponds to minimax (a coalition first tries to maximize its revenues, with the complement set then trying to minimize it). Full definitions and links to our contexts are provided in Appendix, but to summarize, the beta approach corresponds to a coalition always picking first (and is thus sometimes optimistic and sometimes pessimistic depending on the sign of the externalities), while the alpha game provides seemingly unreasonably pessimistic values, in which the complement set of agents goes out of their way to hurt the coalition in question. By opposition, our approach is restricted to credible threats, in which the complement set of agents maximizes its own benefits.

Alternative approaches model strategic interactions between a coalition and the complement set through different equilibrium concepts. For example, Chander and Tulkens (1997) define a value function based on a Nash equilibrium between coalition S and singleton opponents in $N \setminus S$, while Huang and Sjöström (2003) allow $N \setminus S$ to form its optimal partition. Similarly, the recursive core introduced by Kóczy (2007) adopts the idea that deviating agents act in their own self-interest and are free to make their decisions, without determining their partition in advance. How the complement coalition would reorganize itself if a coalition breaks from the grand coalition is a frequent question of interest, with partition function form games (Kóczy, 2018) consigning all such possibilities.

Optimistic and pessimistic assumptions on the behavior of the complement set are common, for example in the coalition formation literature. Our idea that the complement set is using a strategy that is optimal for them, instead of trying to hurt S, is also found in Ray and Vohra (1997). The non-cooperative interplay between S and $N \setminus S$ is found notably in Ichiishi (1981). This strand of literature seeks a solution concept that consistently addresses the strategic effects of externalities in the coalition formation process, whereas our aim is to provide a unified model for TU games based on joint optimization problems, with the ultimate goal of providing fair and stable allocations.

Closer to our perspective, Curiel and Tijs (1991) introduced two operators, minimarg and maximarg, which determine each coalition's marginal contribution based on the worst and the best possible order of agents, respectively. The minimarg assigns the smallest marginal contribution, while the maximarg assigns the largest, embodying pessimistic and optimistic viewpoints, respectively. Iteratively applying these operators to a game leads to a convex and concave game in the limit, with these games being dual to each other. Our approach differs in that they build these operators from a given value game, while we consider the underlying problem of how to define the games themselves.

1.2. Organization of this paper

The paper is organized as follows. Section 2 provides some preliminaries on TU games. Section 3 introduces the framework and defines the optimistic and pessimistic value functions. In Section 4 we provide our main results: (i) an inclusion result between the set of allocations making sure that no coalition gets more than the optimistic upper bounds and the one guaranteeing the pessimistic lower bounds, and (ii) the guaranteed existence of such allocations when feasibility externalities are negative. In Section 5 we apply our framework to a wide range of applications that have been well-studied in the literature. Finally, Section 6 concludes the paper with further extensions.

2. Preliminaries

A *cooperative game with transferable utility* (or TU game) is defined by a pair (N, v) where N is the (finite) set of agents and v is a value function that assigns the value $v(S)$ to each coalition $S \subseteq N$ with $v(\emptyset) = 0$. The number $v(S)$ is the value of the coalition. Whenever no confusion may arise as to the set of players, we will identify a TU game (N, v) with its value function v.

Given a game v, an allocation is a tuple $x \in \mathbb{R}^N$ representing players' respective allotment. The total payoff of a coalition S is denoted by $x(S) = \sum_{i \in S} x_i$ with $x(\emptyset) = 0$. An allocation is *efficient* if $x(N) = v(N)$, and *coalitionally rational* if $x(S) \geq v(S)$ for all $S \subseteq N$.

An allocation is said to be in the *core* of v if it is efficient and coalitionally rational. Then, the core of the game v is the set of all such allocations:

$C(v) = \{x \in \mathbb{R}^N : x(S) \geq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\}$. An allocation is said to be in the *anti-core* of v if it is efficient and for all coalitions the reversed coalitional rationality inequalities hold. Then, the anti-core of the game v is the set of all such allocations: $\mathcal{A}(v) = \{x \in \mathbb{R}^N : x(S) \leq v(S) \text{ for all } S \subset N \text{ and } x(N) = v(N)\}$.

Convexity and concavity (Shapley, 1971) are conditions that have been extensively studied to prove balancedness. A game (N, v) is said to be *convex* if $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$. A game (N, v) is said to be *concave* if $v(T \cup \{i\}) - v(T) \leq v(S \cup \{i\}) - v(S)$ for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$.

The Shapley value (Shapley, 1953) is a single-valued solution that has interesting fairness properties. It is the weighted sum of the agents' marginal contributions to all coalitions. Formally, given a game (N, v), the Shapley value $Sh(v)$ assigns to each agent $i \in N$ the payoff $Sh_i(v) = \sum_{S\subseteq N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} [v(S\cup\{i\}) - v(S)]$.

A game (N, v^*) is the *dual game* of the game (N, v) if $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$.

For dual games, it is well-known that the anti-core of v coincides with the core of v^*, and vice versa.

Proposition 1. *If v and v^* are dual, then $\mathcal{A}(v) = C(v^*)$ and $\mathcal{A}(v^*) = C(v)$.*

3. The model

We build a model that allows for what we call *feasibility externalities*, where the actions of others do not have a direct impact on the revenues one receives, but these actions might affect the set of actions one can take.

Formally, each agent $i \in N$ can take actions, with the set of possible actions defined as $\mathbb{A}_i$. For each agent, the null action $\ominus_i \in \mathbb{A}_i$ means that one possible action is to stay inactive. For each $S \subseteq N$, we define as $\mathbb{A}^S = \bigtimes_{i\in S} \mathbb{A}_i$ the set of actions jointly available to S and $\mathbb{A} \equiv \mathbb{A}^N$.

When agents choose their actions, some actions might not be available. We thus define the *feasible set*, which depends on the actions of other agents. More precisely, for all $S \subsetneq N$ and all $a_{N\setminus S} \in \mathbb{A}^{N\setminus S}$, $f_S(a_{N\setminus S}) \subseteq \mathbb{A}^S$ represents the set of actions jointly feasible for S. We suppose that these sets are always non-empty, since for any coalition, all agents being inactive, $\ominus_S$, is always available as an action. Since the coalition N includes all players, we write f_N instead of $f_N(\ominus_\emptyset)$. Let f represent the set of all such feasibility functions for all coalitions S. We impose a **feasibility complementarity** condition: for all $S \subset N$ and $a_{N\setminus S} \in \mathbb{A}^{N\setminus S}$, $a_S \in f_S(a_{N\setminus S})$ if and only if $(a_S, a_{N\setminus S}) \in f_N$. In words, we assume that if a coalition selects first and the remaining agents select next, the combination of actions is jointly feasible for the grand coalition. Inversely, a set of feasible actions for the grand coalition must be such that if $N \setminus S$ picks their actions in that set first, the remaining actions are feasible for S. The condition is mild and satisfied by all the applications in this paper. Consider the following example that fails the condition: Both S and $N \setminus S$ have a fixed budget to spend, but each dollar spent by $N \setminus S$ decreases the budget for S, while spending by S has no impact on $N \setminus S$. Suppose that S picks first and spends all of its budget, then $N \setminus S$ does the same. Then, the combination of their actions is not feasible for the grand coalition.

For each agent $i \in N$ we have a revenue function $R_i : \mathbb{A}_i \to \mathbb{R}$. Let R represent the set of individual revenue functions. Given that we often have coalitions maximizing their joint revenues, if coalition S chooses the set of actions a_S, we abuse notation and write $R_i(a_S)$ instead of $R_i((a_S)_i)$ for all $i \in S$.

The grand coalition faces an optimization problem that we generally write as $\max_{a_N\in f_N} \sum_{i\in N} R_i(a_N)$. We define a problem P as $(\mathbb{A}, f, R)$, which describes the set of actions, the feasibility sets, and the revenue functions. We suppose that the maximization problem induced by $P = (\mathbb{A}, f, R)$ has a solution. Let $\mathcal{P}$ be the set of all such problems (for all $\mathbb{A}, f, R$).

Example 2. Suppose a simple queueing problem. All agents in N have one single job to be processed on a machine. The machine can process one job per period, and agents have linear waiting costs: if agent i's job is processed in period t, he suffers a cost of $t \times w_i$, where $w_i \geq 0$ is his personal waiting cost parameter.

In this context, we can set $\mathbb{A}_i = \{1, \dots, |N|\}$ to be the set of periods in which i's job could be processed.[2] Then, for any $S \subseteq N$, $f_S(\ominus_{N\setminus S})$ represents what is jointly feasible for S if $N \setminus S$ is inactive, i.e., if their jobs are not processed. We then have that $f_S(\ominus_{N\setminus S})$ is a function $\theta^S : S \to \mathbb{A}^S$ such that $\theta^S_i \neq \theta^S_j$ for all i, j different in S. In words, no two agents in S can be assigned the same processing period.

For $a_{N\setminus S} \neq \ominus_{N\setminus S}$, we have the additional constraint that $\theta^S_i \neq a_j$ for all $i \in S$ and $j \in N \setminus S$. Stated otherwise, the agents in S cannot be assigned to a period already occupied by an agent in $N \setminus S$.

Finally, we have $R_i(a_i) = -w_i a_i$ for all $i \in N$, i.e., each agent has a disutility w_i per period waiting before the job is processed.[3] We can thus rewrite the problem of the grand coalition as $\max_{\theta\in\Theta(N)} \sum_{i\in N} -\theta_i w_i$ where $\Theta(N)$ is the set of bijections from N to $\{1, \dots, |N|\}$.

Example 3. We consider agents that share a joint production technology for a homogeneous good, that they consume in discrete units. The production technology is represented by a non-decreasing function $C : \mathbb{N} \to \mathbb{R}_+$ that assigns a cost to any quantity of good produced, with $C(0) = 0$. For each $k \in N$, $k > 0$, let $c_k = C(k) - C(k-1)$ be the marginal cost of the kth unit. Each agent must decide how much he wants to consume and how much to pay.

In this context we set $\mathbb{A}_i = (q_i, r_i)$ where $q_i \in \mathbb{N}$ is the amount consumed and $r_i \in \mathbb{N}^{q_i}$ assigns to each unit consumed a marginal cost that is paid by the agent. In particular, r_{ik} indicates which marginal cost agent i pays for its kth unit. We write $\ominus_i = (0, \emptyset)$.

Then, for any $S \subseteq N$, $f_S(\ominus_{N\setminus S})$ represents what is jointly feasible for S if $N \setminus S$ is inactive, i.e., if they do not consume any good and pay anything. We then have that $f_S(\ominus_{N\setminus S}) = \{(q_S, \{r_i\}_{i\in S}) \mid \sum_{\substack{i\in S\\ q_i>0}} \sum_{k=1}^{q_i} c_{r_{ik}} \geq C\left(\sum_{i\in S} q_i\right)\}$. In words, f_S is a budget set in which the coalition must collect enough money to cover for the cost of the units it wants to consume. For instance, if we have strictly increasing marginal costs, then coalition S can decide to consume no units and pay nothing, consume one unit and pay any of the marginal costs (as they are all at least as large as c_1), consume two units and pay any two marginal costs (it cannot twice pay c_1, but any other pair is feasible), etc. Of course, if it decides to consume k units, it is optimal to pay for the first k marginal costs, i.e., just cover the cost of the units consumed.

For $a_{N\setminus S} = (q_{N\setminus S}, \{r_i\}_{i\in N\setminus S}) \neq \ominus_{N\setminus S}$, we have that

$$f_S(q_{N\setminus S}, \{r_i\}_{i\in N\setminus S}) = \left\{(q_S, \{r_i\}_{i\in S}) \mid \sum_{\substack{i\in S\\ q_i>0}} \sum_{k=1}^{q_i} c_{r_{ik}} + \sum_{\substack{i\in N\setminus S\\ q_i>0}} \sum_{k=1}^{q_i} c_{r_{ik}} \geq C\left(\sum_{i\in S} q_i + \sum_{i\in N\setminus S} q_i\right)\right\}.$$

In words, the marginal costs paid by agents in S must be enough to cover for the cost of the total number of units consumed, net of what was paid by $N \setminus S$. Note that if $N \setminus S$ just covered the cost of their consumption, then we need $\sum_{\substack{i\in S\\ q_i>0}} \sum_{k=1}^{q_i} c_{r_{ik}} \geq C\left(\sum_{i\in S} q_i + \sum_{i\in N\setminus S} q_i\right) - C\left(\sum_{i\in N\setminus S} q_i\right)$.

Finally, we have $R_i(q_i, r_i) = u_i(q_i) - \sum_{k=1}^{q_i} c_{r_{ik}}$ where $u_i : \mathbb{N} \to \mathbb{R}$ is a non-decreasing utility function such that $u_i(0) = 0$.

3.1. Externalities

We say that a problem exhibits **negative externalities** if for all $i \in S \subsetneq N$ and all $a_{N\setminus S} \in f_{N\setminus S}(\ominus_S)$, we have $f_S(a_{N\setminus S}) \subseteq f_S(\ominus_{N\setminus S})$. Let $\mathcal{P}^-$ be the set of all such problems.

We say that a problem exhibits **positive externalities** if for all $i \in S \subsetneq N$ and all $a_{N\setminus S} \in f_{N\setminus S}(\ominus_S)$, we have $f_S(a_{N\setminus S}) \supseteq f_S(\ominus_{N\setminus S})$. Let $\mathcal{P}^+$ be the set of all such problems.

[2] More precisely, $a_i \in \mathbb{A}_i$ means that the jobs start processing in period $a_i - 1$ and is completely processed at time a_i.

[3] If $a_i = \ominus_i$, then $R_i = -D$, with D arbitrarily large, as the agent's job is not processed.

3.2. Defining cooperative games

It is not always trivial to determine what value to assign to a coalition. In the presence of externalities, the value depends on assumptions we make about the behavior of agents external to the coalition considered.

In many applied problems, as queueing and minimum cost spanning tree problems, a simple approach taken consists of coalition S choosing either first or after $N \setminus S$. If it chooses first, then we have that[4]

$$v^F(S) = \max_{a_S \in f_S(\Theta_{N\setminus S})} \sum_{i\in S} R_i(a_S).$$

If S chooses after $N \setminus S$, we specify that the objective of $N \setminus S$ is not to harm S, but rather to maximize its own revenues. Let $\mu_S = \arg\max_{a_S \in f_S(\Theta_{N\setminus S})} \sum_{i\in S} R_i(a_S)$ be the set of maximizers when S is choosing first. Since the feasible set – and thus the payoff S might receive after $N \setminus S$ has chosen – depends on which maximizer $N \setminus S$ has picked, we define minimum and maximum values as follows:

$$v^L_{\min}(S) = \max_{a_S \in f_S(a_{N\setminus S})} \min_{a_{N\setminus S} \in \mu_{N\setminus S}} \sum_{i\in S} R_i(a_S)$$

and

$$v^L_{\max}(S) = \max_{a_S \in f_S(a_{N\setminus S})} \max_{a_{N\setminus S} \in \mu_{N\setminus S}} \sum_{i\in S} R_i(a_S).$$

In some applications, the maximizer chosen by $N \setminus S$ has no impact on S, in which case we abuse notation and simply write v^L.

We illustrate with the following example.

Example 4. We reconsider Example 3 and now suppose that $N = \{1,2,3\}$ and that the common technology of production exhibits decreasing returns to scale. We describe utility functions by vectors of marginal utilities and the cost function by a vector of marginal costs.

More precisely, agent 1 has marginal utilities of 6 for the first unit, 3 for the second, and zero afterwards. Agent 2 has marginal utility of 12 for the first unit, 6 for the second, and zero afterwards. Agent 3 has marginal utility of 12 for the first unit, 8 for the second, 4 for the third, and zero afterwards. The marginal cost of producing the xth unit is $x-1$.

We obtain the following values for the games we have defined:

S	$v^L_{\min}(S)$	$v^L_{\max}(S)$	$v^F(S)$
$\{1\}$	1	2	8
$\{2\}$	9	9	17
$\{3\}$	11	13	21
$\{1,2\}$	12	12	21
$\{1,3\}$	17	17	24
$\{2,3\}$	24	24	32
$\{1,2,3\}$	34	34	34

We explain some of these values in detail. Starting with v^F, it is obvious that a coalition picking first will always choose to pay for the first K marginal costs if it consumes K units. First, consider $v^F(\{2,3\})$. The coalition chooses first, and faces low marginal costs. If it produces 5 units (2 for agent 2 and 3 for agent 3), it obtains $12-0+12-1+8-2+6-3+4-4=32$. If it produces 4 units (2 for agent 2 and 2 for agent 3), it obtains $12-0+12-1+8-2+6-3=32$. It is easy to see that any other combination yields less net revenues. Thus, $v^F(\{2,3\}) = 32$ but $\mu_{\{2,3\}}$ contains two elements: either the coalition consumes 4 or 5 units, each time paying the first marginal costs.

We now move to v^L_{min} and v^L_{max}. Consider $v^L_{\min}(\{1\})$. We suppose that coalition $\{2,3\}$ has selected an action that maximizes its own revenue, which means that it paid for the first K marginal costs if it consumed K units. For $\{1\}$ the worst maximizer of $\{2,3\}$ is for them to consume 5 units (and pay the first five marginal costs). Agent 1 then faces a marginal cost of 5 on the first unit consumed, 6 on the second, etc. Given that, it is optimal to consume a single unit for a net revenue of $6-5=1=v^L_{\min}(\{1\})$. Moving to $v^L_{\max}(\{1\})$, we now suppose that coalition $\{2,3\}$ has picked its maximizer that is most favorable to agent 1, here consuming 4 units. Thus, agent 1 is now facing a marginal cost of 4 on the first unit consumed, 5 on the second, etc. It is still optimal to consume a single unit, but the net revenue is now $6-4=2=v^L_{\max}(\{1\})$.

The following proposition is immediate.

Proposition 5.

1. *For all $P \in \mathcal{P}^-$ and all $S \subseteq N$, $v^L_{min}(S,P) \le v^L_{max}(S,P) \le v^F(S,P)$.*
2. *For all $P \in \mathcal{P}^+$ and all $S \subseteq N$, $v^F(S,P) \le v^L_{min}(S,P) \le v^L_{max}(S,P)$.*

We can see in the previous result that with negative externalities, choosing first is a favorable scenario, and offers an upper bound, while choosing last is an unfavorable scenario that offers a lower bound. The ranking is flipped with positive externalities. Thus, it is natural to use the following definitions for the optimistic and pessimistic games.

Definition 6.

1. For all $P \in \mathcal{P}^-$, $v^p(\cdot,P) \equiv v^L_{min}(\cdot,P)$ and $v^o(\cdot,P) \equiv v^F(\cdot,P)$.
2. For all $P \in \mathcal{P}^+$, $v^p(\cdot,P) \equiv v^F(\cdot,P)$ and $v^o(\cdot,P) \equiv v^L_{max}(\cdot,P)$.

4. Main results

We provide our main results. We first establish a nice feature of the first/last games. If we use them as lower/upper bounds, then we immediately obtain an inclusion result: the set of allocations making sure that nobody receives more than their upper bounds is a subset of the set of allocations making sure that nobody receives less than their lower bounds.

Theorem 7. *For all $P \in \mathcal{P}$, we have that*

(i) $\mathcal{A}\left(v^L_{max}(\cdot,P)\right) \subseteq C\left(v^F(\cdot,P)\right)$;
(ii) $\mathcal{A}\left(v^F(\cdot,P)\right) \subseteq C\left(v^L_{min}(\cdot,P)\right)$.

Proof. Fix P, and thus write $v^F(S)$, $v^L_{min}(S)$ and $v^L_{max}(S)$ instead of $v^F(S,P)$, $v^L_{min}(S,P)$ and $v^L_{max}(S,P)$. Let $v(N) \equiv \max_{a_N\in f_N} \sum_{i\in N} R_i(a^N)$. Notice that $v^F(N) = v^L_{min}(N) = v^L_{max}(N) = v(N)$.

We start with part **(i)**.

An allocation $x \in \mathcal{A}\left(v^L_{max}\right)$ if $v(N) - v^L_{max}(N\setminus S) \le x(S) \le v^L_{max}(S)$ for all $S \subseteq N$. An allocation $x \in C\left(v^F\right)$ if $v^F(S) \le x(S) \le v(N) - v^F(N\setminus S)$ for all $S \subseteq N$. It is easy to see that $\mathcal{A}\left(v^L_{max}\right) \subseteq C\left(v^F\right)$ if and only if $v(N) \ge v^L_{\max}(S) + v^F(N \setminus S)$ for all $S \subseteq N$. Fix coalition $S \subseteq N$ and let a^*_N be (one of) the optimal set(s) of actions taken by N. Then, $v(N) = \sum_{i\in N} R_i(a^*_N)$, $v^F(N\setminus S) = \sum_{i\in N\setminus S} R_i\left(a^+_{F(N\setminus S)}\right)$ and $v^L_{max}(S) = \sum_{i\in S} R_i\left(a^+_{L(S)}\right)$, where $a^+_{F(N\setminus S)}$ is the best maximizer for S among the maximizers when $N\setminus S$ chooses first and $a^+_{L(S)}$ is a maximizer for S after $N\setminus S$ has chosen $a^+_{F(N\setminus S)}$. We thus have that

$$\begin{aligned} v^L_{max}(S) + v^F(N\setminus S) &= \sum_{i\in S} R_i\left(a^+_{L(S)}\right) + \sum_{i\in N\setminus S} R_i\left(a^+_{F(N\setminus S)}\right) \\ &= \sum_{i\in N} R_i\left(\{a^+_{L(S)}, a^+_{F(N\setminus S)}\}\right) \\ &\le \sum_{i\in N} R_i(a^*_N) \\ &= v(N), \end{aligned}$$

where the inequality follows from the fact that, by feasibility complementarity, $\{a^+_{L(S)}, a^+_{F(N\setminus S)}\} \in f_N$.

Next, we show **(ii)**. An allocation $x \in \mathcal{A}\left(v^F\right)$ if $v(N) - v^F(N\setminus S) \le x(S) \le v^F(S)$ for all $S \subseteq N$. An allocation $x \in C\left(v^L_{min}\right)$ if $v^L_{min}(S) \le$

[4] We suppose, in this problem and in subsequent ones, that the optimization problem for coalition $S \subset N$ has a solution.

$x(S) \leq v(N) - v^L_{min}(N \setminus S)$. It is easy to see that $\mathcal{A}\left(v^F\right) \subseteq C\left(v^L_{min}\right)$ if and only if $v(N) \geq v^F(S) + v^L_{min}(N \setminus S)$ for all S. The results follows from the proof of part (**i**) and Proposition 5. □

Given our definitions of the optimistic and pessimistic value functions, we obtain the following corollary: guaranteeing that no coalition receives more than its upper bound is at least as difficult as guaranteeing that it receives at least its lower bound.

Corollary 8. *For all $P \in \mathcal{P}^- \cup \mathcal{P}^+$, we have that $\mathcal{A}(v^o(\cdot, P)) \subseteq C(v^p(\cdot, P))$.*

Next, we show that when we have negative externalities the anti-core of the optimistic game is always non-empty.

Theorem 9. *For all $P \in \mathcal{P}^-$, $\mathcal{A}(v^o(\cdot, P))$ is non-empty.*

Proof. Fix $P \in \mathcal{P}^-$ and write $v^F(S)$ and $v^o(S)$ instead of $v^F(S,P)$ and $v^o(S,P)$. Since $P \in \mathcal{P}^-$ we have that $v^o = v^F$.

Let $S \subset N$ and let a^* be (one of) the maximizer(s) for the problem of the grand coalition and $a_{F(S)}$ be one of the maximizers when S selects first. We show that $v^o(S) \geq \sum_{i \in S} R_i(a^*)$.

We have that

$$\begin{aligned} v^o(S) &= \sum_{i \in S} R_i\left(a_{F(S)}\right) \\ &\geq \sum_{i \in S} R_i\left(a^*_S\right) \\ &= \sum_{i \in S} R_i\left(a^*\right), \end{aligned}$$

where the inequality is by definition of $a_{F(S)}$, since $a^*_S \in f_S(\ominus_{N \setminus S})$.

Since a^* is a maximizer for the grand coalition, by definition $\sum_{i \in N} R_i(a^*) = v^o(N)$. Thus, $\left(R_i(a^*)\right)_{i \in N}$ is in $\mathcal{A}(v^o)$. □

Combining our two main results, we obtain the following corollary.

Corollary 10. *For all $P \in \mathcal{P}^-$, $\emptyset \neq \mathcal{A}(v^o(\cdot, P)) \subseteq C(v^p(\cdot, P))$.*

Thus, with very little structure on the problem other than negative feasibility externalities, we are able to show the non-emptiness of the core of the pessimistic game. Negative externalities lead to substantial benefits from cooperation to improve efficiency, and we can always distribute these benefits in a stable manner.

On the other hand, the guarantee of a non-empty anti-core does not carry over to problems with positive externalities, as illustrated in the counterexamples below. This shows the (possibly) counterintuitive results that negative externalities yield inherently stable games, while positive externalities can lead to instability as there is more temptation to free ride.

Example 11. We modify Example 4 to suppose increasing returns to scale in production. Suppose the same marginal utilities, but now the marginal cost of production is 14 for the first unit, 9 for the second, 7 for the third, 3 for the fourth and 1 afterwards.

We obtain the following values:

S	$v^p(S)$	$v^o(S)$
$\{1\}$	0	7
$\{2\}$	0	0
$\{3\}$	0	0
$\{1,2\}$	0	0
$\{1,3\}$	0	0
$\{2,3\}$	8	8
$\{1,2,3\}$	15	15

We explain how the values for coalitions $\{1\}$ and $\{2,3\}$ are computed. If agent 1 has to choose first, it faces too high marginal costs, and it consumes nothing, and $v^p(\{1\}) = 0$. Coalition $\{2,3\}$, acting first, will consume 5 units to generate a net surplus of $12 - 14 + 12 - 9 + 8 - 7 + 6 - 3 + 4 - 1 = 8 = v^p(\{2,3\})$.

Now, consider $v^o(\{1\})$. We have that coalition $\{2,3\}$ has consumed 5 units so agent 1 is now facing marginal costs of 1. He consumes 2 units for a gain of $6 - 1 + 3 - 1 = 7 = v^o(\{1\})$. Since $\{1\}$, when alone, does not consume, we have that $v^o(\{2,3\}) = v^p(\{2,3\}) = 8$.

To find an allocation in $\mathcal{A}(v^o)$, we need $y_1 \leq 7$, $y_2 \leq 0$ and $y_3 \leq 0$, which are incompatible with $y_1 + y_2 + y_3 = 15$, and thus $\mathcal{A}(v^o) = \emptyset$. Notice that here $C(v^p) \neq \emptyset$.

Example 12. Atay and Trudeau (2024b) provide a variant of the queueing problem by supposing that agents must buy machines to queue on, and can buy as many machines as they want. The problem becomes one with positive feasibility externalities: by itself, a coalition can only buy its own machines and queue on them; if it joins others, it can still do so, but can also take advantage of unused time slots on their machines. Hence, in this case, choosing last corresponds to the optimistic approach, $v^o = v^L_{max}$. Atay and Trudeau (2024b) show that the core of the corresponding pessimistic game is sometimes empty, sometimes not. By Theorem 7, so is the anti-core of the optimistic game.

The use of both an optimistic and a pessimistic game yields two (potentially) interesting Shapley values. Their stability depends on the convexity/concavity of these games.

Proposition 13. *For all $P \in \mathcal{P}^- \cup \mathcal{P}^+$ we have:*

1. *if $v^o(\cdot, P)$ is concave then $Sh(v^o(\cdot, P)) \in \mathcal{A}(v^o(\cdot, P)) \subseteq C(v^p(\cdot, P))$;*
2. *if $v^p(\cdot, P)$ is convex then $Sh(v^p(\cdot, P)) \in C(v^p(\cdot, P))$.*

This increases the chances of finding a stable allocation: if we are interested in v^p, but it is not convex, we have a backup: if v^o is concave its Shapley value is in the core of the pessimistic game.

5. Applications

In this section, we discuss several applications that exhibit feasibility externalities. We examine how these applications can be modeled within our framework, how optimistic and pessimistic approaches have been defined in each case, and whether our results allow to reinterpret existing results.

5.1. Queueing problems

We first examine more formally our example of queueing problems. Consider a set of agents N that each have a job to be processed at one machine. The machine can process only one job at a time. Each agent $i \in N$ incurs waiting costs $w_i > 0$ per unit of time. The queueing problem determines both the order in which to serve agents and the corresponding monetary transfers they should receive (see Chun (2016) for a survey on the queueing problem). See Example 2 for the definition of the problem in our framework.

These pessimistic and optimistic approaches have been defined independently in the literature. Maniquet (2003) built the optimistic game, using the assumption that a coalition is served before the players outside the coalition. The minimal transfer rule,[5] ϕ^{min}, is obtained by applying the Shapley value to v^o. Alternatively, Chun (2006) assumes that a coalition is served after the non-coalitional members, obtaining the pessimistic game.[6] The maximal transfer rule,[7] ϕ^{max}, is obtained by applying the Shapley value to v^p.

We obtain the following results.

[5] The minimal transfer rule assigns to each agent a position in the queue and a monetary transfer. The monetary transfer is equal to half of their unit waiting cost multiplied by the number of agents in front of them in the queue minus half of the sum of the unit waiting costs of the people behind them in the queue.

[6] Independently, Klijn and Sánchez (2006) considered the same scenario as in Chun (2006). They introduced the associated game, the so-called tail game, and studied its core.

Theorem 14. *For any queueing problem, we have $\phi^{min} \in \mathcal{A}(v^o) \subseteq \mathcal{C}(v^p)$ and $\phi^{max} \in \mathcal{C}(v^p)$.*

The results that v^o is concave and $\phi^{min} \in \mathcal{A}(v^o)$ is easily obtained from Maniquet (2003), while the results that v^p is convex and $\phi^{max} \in \mathcal{C}(v^p)$ come from Chun (2006). The result that $\phi^{min} \in \mathcal{C}(v^p)$ can be obtained by comparing the shares of a coalition to its lower bound; but by our Proposition 13, the result is obtained without any further calculations. Hence, we obtain a new justification for the minimal transfer rule. While both rules offer allocations above the pessimistic bounds, the minimal transfer allow guarantees allocations below the optimistic bounds.

5.2. Minimum cost spanning tree problems

We have a set of nodes consisting of $N_0 \equiv N \cup \{0\}$, where 0 is a special node called the source. Agents need to be connected to the source to obtain a good or a service. To each edge $(i,j) \in N_0 \times N_0$ corresponds a cost $c_{ij} \geq 0$, with the assumption that $c_{ij} = c_{ji}$. These costs are fixed costs, paid once if an edge is used, regardless of how many agents use it. The problem is to connect all agents to the source at the cheapest cost. Given the assumptions above, among the optimal networks there always exists a spanning tree, hence the name of the problem. A minimum cost spanning tree (mcst) problem is (N,c), where c is the list of all edge costs. c is also called a cost matrix.

The set of actions of player i is the set of edges containing node i: agent i chooses $j \in N_0 \setminus i$, building the edge (i,j). The usual assumption is to suppose that a coalition S cannot use edge (i,j) if $k \in \{i,j\}$ is such that $k \in N \setminus S$. Then $f_S(\ominus_{N\setminus S})$ is the set of spanning trees rooted at 0 that does not use nodes in $N \setminus S$, while in $f_S(a_{N\setminus S})$, for any $a_{N\setminus S}$, we also treat agents $i \in N \setminus S$ such that $(a_{N\setminus S})_i \neq \ominus_i$ as additional sources. Thus, we obtain a problem with positive feasibility externalities. We complete the representation in our setting by having $R_i = -c_{ia_i}$, i.e., agent i pays for the cost of the edge he builds.

Most of the literature has considered the pessimistic game v^p, in which a coalition S connects to the source first, before $N \setminus S$. An exception is Bergantiños and Vidal-Puga (2007b), which considers the game in which coalition S supposes that $N \setminus S$ has already connected to the source. In such a case, agents in $N \setminus S$, being connected, act as sources for S. Thus, how they are connected is irrelevant, and this game is equivalent in our notation to both v^L_{min} and v^L_{max}.

The literature has devoted considerable attention to the notion of irreducible cost matrix (Bergantiños & Vidal-Puga, 2007a; Feltkamp et al., 1994): since many edges are not used in any optimal spanning tree, we reduce the cost of these edges as much as possible, under the constraint that $v^p(N)$ does not change. There is a unique way to do so, and irreducible edge costs can be obtained as follows: take any optimal spanning tree, and for each pair of nodes $(i,j) \in N_0$, look at the (unique) path from one to the other, and assign to (i,j) the most expensive edge on that path. We then obtain the irreducible cost matrix $\hat{c}$. Let $\hat{v}^p$ and $\hat{v}^o$ be the pessimistic and optimistic games obtained from the irreducible cost matrix.

Theorem 15 (*Bergantiños & Vidal-Puga, 2007b*). *For any mcst problem (N,c), we have*

(i) $\hat{v}^p$ and $\hat{v}^o$ are dual.
(ii) $\hat{v}^o = v^o$.

This leads us, using our results, to the following corollary.

Corollary 16. *For any mcst problem (N,c) we have $\mathcal{A}(v^o) = \mathcal{C}(\hat{v}^p)$.*

This result is interesting for three reasons. First, $\mathcal{C}(\hat{v}^p)$ is called the irreducible core (Bird, 1976), and has been shown to be uniquely characterized by additivity and monotonicity properties (Bergantiños & Vidal-Puga, 2015; Tijs et al., 2006). Second, our equivalence with the anti-core of the optimistic game means that we do not need to go through the modification of the cost matrix into the irreducible matrix to obtain the irreducible core. Third, since the game is concave (Bergantiños & Vidal-Puga, 2007a), the Shapley value is guaranteed to be in the core of the pessimistic game. This result follows directly from Proposition 13, without the need to analyze the problem under its specifications. The resulting allocation rule is the well-studied Folk rule.

It is also worth noting that the allocation used to prove the non-emptiness of the anti-core of the optimistic game in Theorem 9 corresponds to the Bird allocation (Bird, 1976) in which each agent pays the cost of the edge connecting it to its nearest neighbor in its unique path to the source.

5.3. River sharing problems

Suppose a river described as a line with agents i being upstream of agent j if and only $i < j$. There is an entry $e_i \geq 0$ of water at each location i, and the water that flows at location i can be consumed by agent i or allowed to flow downstream. The benefit from water consumption for agent i is given by a strictly increasing and strictly concave function b_i such that $b_i(0) = 0$. A water sharing problem is (N,e,b), with the set of players N, the vector of water entries e, and the collection of benefit functions b (Ambec & Sprumont, 2002). The problem for the grand coalition is to maximize joint benefits, under the constraint imposed by the flows of water. If $x_i \geq 0$ is the consumption level of agent i, the feasible set is constrained as follows: for any $i \in N$, that $\sum_{j\leq i} x_j \leq \sum_{j\leq i} e_j$. For a coalition S, if the complement set is consuming any amount of water, the feasible set is reduced, and we thus have negative feasibility externalities. Thus, $v^o = v^F$ and $v^p = v^L_{min}$ and by Theorem 9, we already know that $\mathcal{A}(v^o)$ and $\mathcal{C}(v^p)$ are non-empty.

If coalition S chooses first, it has access to all water entries in the river, subject to the physical constraints imposed by the river, i.e., an agent upstream of a location cannot consume the water entry at that location. Thus, we obtain that $v^F(S) = \max_{(x_i)_{i\in S}} \sum_{i\in S} b_i(x_i)$ under the constraints that $\sum_{\substack{j\leq i\\ j\in S}} x_j \leq \sum_{j\leq i} e_j$ for all $i \in S$.

If S chooses last, then $N\setminus S$, given that its members are not satiable, have consumed as much water as they could. The exact maximizer is thus irrelevant, and we have $v^L \equiv v^L_{min} = v^L_{max}$. To define v^L properly we need the following definition: a coalition is consecutive if for any pair of agents in that coalition, adjoining agents are also in the coalition. Thus, we have that $v^L(S) = 0$ if $n \notin S$ and $v^L(S) = \max_{(x_i)_{i\in S^n}} \sum_{i\in S^n} b_i(x_i)$ under the constraints that $\sum_{\substack{j\leq i\\ j\in S^n}} x_j \leq \sum_{\substack{j\leq i\\ j\in S^n}} e_j$ for all $i \in S^n$ otherwise, where S^n is the largest consecutive coalition in S that contains n. In words, if $i \in S$ is such that a member of $N \setminus S$ is downstream, then the water entries at i and upstream have all been consumed by $N \setminus S$. Thus, the only group in S that is able to consume is S^n, such that all its members are downstream of all members of $N \setminus S$.

The coalitional functions proposed in the literature have been constructed from various doctrines used in international law. Under the unlimited territorial integrity (UTI) doctrine, an agent can consume any water that flows through its location. $v^{UTI}(S)$ is seen as an upper bound on the welfare of S, and it is easy to see that it corresponds to v^F.

Under the absolute territorial sovereignty (ATS) doctrine, an agent has absolute rights over the water entering on its territory. For a single agent i, this implies that he should received at least $b_i(e_i)$. For larger coalitions, we suppose that an agent i can transfer water to j only if j is its immediate downstream neighbor. Otherwise, the water is

[7] The maximal transfer rule assigns to each agent a position in the queue and a monetary transfer. The monetary transfer is equal to half of the sum of the unit waiting costs of her predecessors minus half of her unit waiting cost multiplied by the number of her followers.

consumed by the agent(s) between i and j. Thus, let $\Gamma(S)$ be the coarsest partition of S into consecutive coalitions. Then, $v^{ATS}(S) = \sum_{T\in\Gamma(S)} \max_{(x_i)_{i\in T}} \sum_{i\in T} b_i(x_i)$ under the constraints that $\sum_{\substack{j\leq i\\ j\in T}} x_j \leq \sum_{\substack{j\leq i\\ j\in T}} e_j$ for all $i \in T$ and all $T \in \Gamma(S)$.

Given the pessimistic constraints in the ATS version of the problem, $v^{ATS}(S)$ is seen as a lower bound on the welfare of S. But it is immediate that $v^{ATS} \geq v^L$, with, in particular, that $v^{ATS}(S) = v^L(S)$ if S is a consecutive coalition containing n and $v^{ATS}(S) \geq 0 = v^L(S)$ if S does not contain n. Thus, while pessimistic, v^{ATS} is much less pessimistic than v^L. We thus obtain that:

$$v^p \equiv v^L \leq v^{ATS} \leq v^{UTI} = v^F \equiv v^o.$$

Ambec and Sprumont (2002) define the downstream incremental allocation rule as follows: $y_i^{DI} = v^{UTI}(\{1,\dots,i\}) - v^{UTI}(\{1,\dots,i-1\}) = v^{ATS}(\{1,\dots,i\}) - v^{ATS}(\{1,\dots,i-1\})$. They show that it is the unique intersection of $\mathcal{A}(v^{UTI}) \cap \mathcal{C}(v^{ATS})$. Trivially, the downstream incremental allocation is also in the core of our pessimistic game.

The use of v^{UTI} as an upper bound as been criticized (Herings et al., 2007; van den Brink et al., 2007), as the most upstream agents can never do better than consume all of their water; the upper bound prevents them from extracting benefits of letting downstream agents consume some of their water. Given Corollary 8 which guarantees that $\mathcal{A}(v^o) \subseteq \mathcal{C}(v^p)$, our results provide an immediate resolution to this problem: there are stable allocations in $\mathcal{C}(v^p)\setminus\mathcal{A}(v^o)$ that allow for larger compensations of upstream agents.

In this model v^p is a particularly pessimistic function, and only coalitions containing n can guarantee a strictly positive lower bound. The function v^{ATS}, which offers higher lower bounds, provides a compromise between the optimistic (UTI) function and the pessimistic function.

Our results (the proof of Theorem 9) also provide an allocation in both $\mathcal{A}(v^o)$ and $\mathcal{C}(v^p)$, a no-transfer rule in which each agent simply consumes its efficient amount of water. While it is particularly beneficial to downstream agents who do not have to compensate upstream agents for their reduced consumption, it has the advantage of being simple to compute: if the counterfactuals are difficult to determine, making the calculation of any other allocation complex or impossible, not imposing any transfers guarantees a stable allocation.

Many extensions of the model have been considered, including to cases where some agents can be satiated (Ambec & Ehlers, 2008) and cases with multiple springs and bifurcations (Khmelnitskaya, 2010). See Béal et al. (2012) for a review.

5.4. Applications with dual game structure

The applications and examples we have examined so far have all been such that studying the problem using lower and upper bounds gave different sets of allocations, and thus the perspective taken mattered. However, in some other cases, the two approaches generate dual games, and as seen in Proposition 1, it is then unnecessary to study v^o and v^p separately. We discuss this duality and show that this duality can be easily spotted.

In our framework, v^o and v^p are dual games if and only if for any coalition S, letting S pick first and $N\setminus S$ react to that afterwards always leads to an efficient outcome. In other words, an optimal outcome can always be obtained by sequential selfish optimizations by S and $N \setminus S$.

Proposition 17. *For a problem $P \in \mathcal{P}^- \cup \mathcal{P}^+$, we have that v^o and v^p are dual games if and only if for all $S \subset N$ there exists a_S and $a_{N\setminus S}$ such that $v^o(S,P) = \sum_{i\in S} R_i(a_S)$, $v^p(N \setminus S, P) = \sum_{i\in N\setminus S} R_i(a_{N\setminus S})$ and $\{a_S, a_{N\setminus S}(a_S)\} \in \arg\max_{a_N\in f_N} \sum_{i\in N} R_i(a_N)$.*

We then have that $\mathcal{A}(v^o) = \mathcal{C}(v^p)$.

Two important applications exhibiting duality are bankruptcy (claims) problems and airport problems.

The *bankruptcy problem* deals with sharing an estate E of a perfectly divisible resource among agents N who have conflicting claims. That is, the sum of claims is larger than the estate: $\sum_{i\in N} c_i > E$ where c_i is the claim of agent i. O'Neill (1982) studied such problems from an economic point of view. He introduced an associated TU game to each bankruptcy problem and also defined the run-to-the-bank rule based on an average over all possible orders on agents' arrival. Within our framework, the action of agent i is the amount that he takes from the estate, which is also his revenue. We thus have a negative feasibility externality problem.

The optimistic approach corresponds to a bank-run situation, in which coalition S arrives first and collects its combined claim or the endowment, whichever is smallest. The pessimistic approach has coalition S arriving last, collecting what is left after the bank run of $N\setminus S$. The combination of the optimistic action of S and the pessimistic action of $N \setminus S$ always leads to a full distribution of the endowment, and thus to an efficient outcome. Following Proposition 17, the two games are dual.

The *airport problem* introduced by Littlechild and Owen (1973) aims to allocate the cost of a landing strip among users with varying runway length requirements. Every agent i requires a length l_i at the runway. It is assumed that the cost to build the runway is non-decreasing in its length. That is, for any two agents i and j such that $l_i < l_j$, $c(l_i) \leq c(l_j)$. Within our framework, the action of agent i is the segment of the runway that he builds, and his revenues is the cost of that segment.[8] We thus have a problem with positive feasibility externalities.

The pessimistic approach assumes that coalition S arrives first to build its runway. The longest runway required by a member of the coalition will be built, which is $\max_{i\in S} l_i$. Suppose next that coalition $N \setminus S$ picks last. Knowing that a runway of length $\max_{i\in S} l_i$ has been built, it extends it, if needed, to a length of $\max_{i\in N} l_i$. Given the inelastic demands of our agents, the length of the runway is efficient. Hence, the optimistic and pessimistic approaches are dual for airport problems.

Recently, Bergantiños and Moreno-Ternero (2025b) studied the distribution of revenues in streaming platforms. They show that the optimistic and the pessimistic approaches leads to dual games, and consequently, their Shapley values coincide. Based on the problem's specifications, Proposition 17 can be used to obtain their result.

We conclude this section by an illustration of the line between duality and non-duality. In a cooperative production problem, a set of agents share a production technology to produce some good(s). This joint production technology might exhibit increasing or decreasing returns to scale/scope.

As shown in Examples 4 and 11, when the quantities consumed are endogenously determined (see, for instance, Fleurbaey & Maniquet, 1996; Moulin, 1990; Roemer & Silvestre, 1993), the approach chosen matters, and a coalition S choosing first and the complement $N \setminus S$ might very well lead to an inefficient quantity being produced and/or its distribution to agents being inefficient. Thus, we have no duality.

If we suppose that demands for the good(s) are exogenous (see, for instance, de Frutos, 1998; Moulin, 1996; Moulin & Shenker, 1992), then producing these inelastic demands is efficient. Given that demands are inelastic, we always obtain that the same (efficient) total quantity is produced by having S and $N \setminus S$ sequentially choose. Thus, the optimistic and pessimistic games are dual.

[8] More precisely, as in minimum cost spanning tree problems, an agent chooses to connect to an agent or the source. If agent i picks $j < i$, he would be responsible for the cost $c(i) - c(j)$. If he picks $j > i$, the cost is zero. For coalition S, the feasible set is such that the combination of choices made by its members must be such that all agents end up connected to the source.

6. Concluding remarks

This paper develops a general framework for analyzing operations research games when the actions of players outside a coalition influence what that coalition can achieve. By capturing how the feasible actions of a coalition depend on the behavior of outsiders, our approach unifies and extends existing modeling assumptions used across a wide range of applications. The theoretical tools and results we provide establish a basis for systematically analyzing new and more complex cooperative situations involving external effects (e.g., pipeline externalities problem (Trudeau & Rosenthal, 2025) and streaming platform (Bergantiños & Moreno-Ternero, 2025c). Moreover, the framework is readily applicable to a wide range of optimization-based problems, including queueing, minimum cost spaning tree, and resource sharing problems where externalities are inherent. We believe this foundation can guide the development of tailored models in future research and facilitate the implementation of cooperative solutions in practical settings, ultimately broadening the scope and impact of cooperative game theory in operations research.

Moreover, our model can be extended to address more complex settings. We have considered what we call feasibility externalities, in which the actions taken affect the feasible sets of other agents. First, when we have direct externalities, in which the actions taken by a group directly affect the revenues obtained by other agents, the additional difficulty is that we have to determine who to credit for these direct externalities. The main message from our results is that there is considerable benefit in defining the optimistic and pessimistic values in a consistent manner: if the combination of what S chooses in the optimistic game and what $N \setminus S$ chooses in the pessimistic game is feasible for the grand coalition N, then we obtain the inclusion result of Theorem 7, even in the presence of direct externalities. The presence of direct externalities yields multiple definitions of optimistic and pessimistic functions.

Another extension involves exploring a wider set of coalition behaviors. For instance, we may ask: while having a coalition move first or last provides natural bounds, are these always the true lower and upper bounds? As they lie beyond the scope of the present paper, for their conceptualization and further results we refer to Atay and Trudeau (2024a). The analysis there shows that the "move first/move last" assumption corresponds to worst- and best-case scenarios in richer strategic settings. Under natural conditions, these extremes reflect the actual boundaries of what coalitions can expect, meaning the simplifications often adopted in practice are not only computationally convenient but also theoretically sound.

CRediT authorship contribution statement

Ata Atay: Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization. **Christian Trudeau:** Writing – review & editing, Writing – original draft, Formal analysis, Conceptualization.

Appendix. Comparison to alpha and beta games

We first examine the classic definitions of α and β games (Shubik, 1982; pp. 136–138). Firstly, α games are maximin: the complement $N \setminus S$ first tries to minimize the payoff to S, and then S acts to maximize its payoff. By opposition β games are minimax: S tries to maximize its payoff, and then the complement $N \setminus S$ tries to minimize their payoffs. In our context, they are defined as follows:

$$v^{\alpha}(S) = \max_{a_S \in f_S(a_{N\setminus S})} \min_{a_{N\setminus S} \in f_{N\setminus S}(\Theta_S)} \sum_{i\in S} R_i(a_S)$$

and

$$v^{\beta}(S) = \min_{a_{N\setminus S} \in f_{N\setminus S}(a_S)} \max_{a_S \in f_S(\Theta_{N\setminus S})} \sum_{i\in S} R_i(a_S) = \max_{a_S \in f_S(\Theta_{N\setminus S})} \sum_{i\in S} R_i(a_S)$$

where the last simplification is because of our assumption that there is no direct externalities. Thus, in v^{β}, $N \setminus S$ cannot do anything. However, in v^{α}, $N \setminus S$ can reduce the value of S if its actions allows to reduce the feasible set of coalition S.

It is immediate that $v^{\beta} = v^F$. Notice that the definitions of v^{α} and $v^L_{\min}$ are very similar, with the only distinction being that we suppose for $v^L_{\min}$ that $N \setminus S$ can only select from its set of maximizers, while in v^{α} it can select any actions from its feasible set. Thus, the distinction is similar to restricting to credible threats. $v^L_{\max}$ is more optimistic in that we suppose that $N \setminus S$ selects its maximizer that is most favorable to S.

Example 18. We revisit Example 4. As discussed, v^{β} corresponds to v^F. Here, v^{α} is particularly simple: for each coalition S there exists a quantity consumed by $N \setminus S$ such that it raises the marginal costs so much that it becomes optimal for S to consume nothing. Thus, $v^{\alpha}(S) = 0$ for all $S \neq N$.

The previous example shows that v^{α} is not interesting because without the constraint of selecting a maximizer, we obtain values that are too low. In other words, v^{α} is too pessimistic as it considers the complement set taking non-credible actions.

Notice that when we have positive externalities, the worst that $N \setminus S$ can do to S is to stay inactive, keeping the feasible set of S as small as possible, and thus $v^{\alpha}(S,P) = v^{\beta}(S,P) = v^F(S,P)$.

We state these observations formally.

Proposition 19.

1. *For all $P \in \mathcal{P}^-$ and all $S \subseteq N$, we have $v^{\alpha}(S,P) \leq v^{p}(S,P) \leq v^{o}(S,P) = v^{\beta}(S,P)$.*
2. *For all $P \in \mathcal{P}^+$ and all $S \subseteq N$, we have $v^{\alpha}(S,P) = v^{\beta}(S,P) = v^{p}(S,P) \leq v^{o}(S,P)$.*

Overall, we can see that v^{β} corresponds to v^F, and thus is not consistently optimistic or pessimistic in our framework. v^{α} is too pessimistic in problems with negative externalities as it allows the complement set to take non-credible actions.

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