On the vertices of the core of a many-to-one assignment game*

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Abstract

We study the structure of the core in many-to-one assignment games, where firms with limited capacity hire workers in a transferable utility framework. While the core in such games is known to be non-empty and admits side-optimal elements, less is known about its full geometry, particularly the characterization and enumeration of its extreme points. We provide a graph-theoretic criterion for core vertices: a salary vector is a vertex of the core if and only if its associated tight digraph is connected. Building on this, we develop a lexicographic procedure that generates all core vertices as they are supported by a max-min salary vector.

Keywords: Many-to-one assignment markets extreme core allocations · side-optimal allocations · core

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1 Introduction

Many-to-one assignment games model two-sided market situations particularly relevant for labor markets and institutional settings where organizations (firms) hire multiple agents (workers), but each agent can engage with only one organization. These models

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extend the classical one-to-one assignment game of Shapley and Shubik (1971), which was originally developed to understand two-sided markets with monetary transfers and indivisible goods. In their model, each buyer wants at most one unit of good, and each seller owns exactly one indivisible good.[1] In the one-to-one case, a comprehensive theory has been developed around the core, that is, the set of allocations of the total value of the market that cannot be improved upon by any coalition, and its connections to competitive equilibrium. One-to-one assignment market games can be extended to many-to-many assignment markets in two different ways. The first one is known as the multiple-partners assignment game (Sotomayor, 1992) and each agent can establish several partnerships, as many as its capacity allows, but each of them with a different partner. In the second extension, sometimes known as the transportation game (Sánchez-Soriano et al., 2001), an agent may establish more than one partnership with a same agent of the opposite side. In both cases the core is proved to be non-empty. Notice that our many-to-one situation lies in the intersection of these two extensions since the unitary capacity of agents on one side rules out the possibility of more than one partnership between any specific firm-worker pair.

Most of the existing results for one-to-one assignment markets cannot be extended to the many-to-many assignment markets (Sotomayor, 2002; Sotomayor, 2007): some core allocations may not be supported by competitive prices and the core may not be a lattice. However, for the many-to-one assignment games, the core retains desirable properties such as non-emptiness and a lattice structure based on the partial order on the set of payoffs to the side of the market where agents have unitary capacity. This guarantees in that case the existence of an optimal core allocation for each side of the market. While classical results have focused on the existence and side-optimality of core allocations, less attention has been paid to the full set of the vertices of the core polytope. In this paper, we study the structure of the core polytope in many-to-one assignment games.

We can consider two types of many-to-one assignment markets, depending on which side of the market has unitary capacity. This does not affect the core, but makes a big difference when competitive equilibria are considered. If the unitary capacity is on the side that posts prices, that is sellers or workers, then we are in the many-to-one model of Sotomayor (2002) and the core coincides with the set of competitive equilibrium payoff vectors. When the unitary capacity is on the agents that report a demand given some prices, that is, buyers or firms, then we are in the many-to-one model of Kaneko (1976), and the core may strictly contain the set of competitive equilibrium payoff vectors (CE payoff vectors) that now coincides with the set of solutions of the dual linear program that finds an optimal matching. In this paper we focus mainly on the first case, the job market with unitary capacity workers of Sotomayor (2002) and hence we indistinctly refer to "core payoff vectors" and "competitive equilibrium payoff vectors". Only in the last section we show that, after some adjustments, parallel results can be obtained to describe the extreme core points of Kaneko's buyer-seller market, where buyers have unitary capacity.

Besides the one-to-one assignment game, the literature contains results regarding the

[1](Núñez and Rafels, 2015) is a survey on assignment markets and games.

set of extreme core allocations for other related combinatorial models, such as ordinal two-sided markets (Baïou and Balinski, 2000, 2002) and minimum cost spanning tree games (Trudeau and Vidal-Puga, 2017).

The primary goal of this paper is to provide a graph-theoretic characterization of the vertices of the core in many-to-one assignment games. Our approach introduces the concept of a tight digraph associated with each core salary vector. This generalizes insights from earlier work on assignment games, where core extremality could be related to some graph properties (Hamers et al., 2002). The proofs of some previous results on the one-to-one assignment game, such as the algorithm to compute the nucleolus in Solymosi and Raghavan (1994) and the characterization of the core stability in Solymosi and Raghavan (2001), also rely on the structure of the underlying bipartite graph.

Based on the projection of the core to the space of workers' payoffs (salaries), and given a competitive salary vector, we define the tight digraph where the set of nodes is the set of workers augmented by a node representing their outside option and the directed arcs are determined by the constraints of the set of competitive salaries that are tight at that given vector. Then, we show that a competitive salary vector is an extreme point if and only if the base-graph of the tight digraph (where the direction of the arcs are ignored) is connected (Theorem 6 (A)). It implies that at an extreme competitive salary vector there is a worker with zero salary or a worker with a salary that equals the total surplus it creates with a firm under an optimal matching. We also provide a necessary and sufficient condition for each side-optimal allocation in terms of the tight digraph (Theorem 6 (B)).

After that, for each order on the set of workers, we define a payoff vector where each worker sequentially maximizes or minimizes its competitive salary, preserving what has been allocated to its predecessors. Making use of the tight digraph, we show that this set of max-min vectors includes all the extreme competitive salary vectors of the many-to-one assignment market. This gives a procedure for the computation of these extreme points and consequently allows for a representation of the entire core.

Before concluding, we move to the other many-to-one assignment market (Kaneko, 1976), let us say a buyer-seller market where buyers have unitary demand. Our results on the core trivially apply to this case, simply focusing on the projection of the core to the buyers' payoffs. We also provide a description of the competitive equilibrium payoff vectors that allows for a characterization of their extreme points by means of an extended tight graph.

The rest of the paper is organized as follows. Section 2 provides preliminaries on cooperative TU games. Section 3 formulates many-to-one assignment markets and games. Section 4 presents a characterization for any extreme competitive salary vector making use of the tight digraph. Based on this, we describe a lexicographic procedure to obtain all extreme vectors of competitive salaries, or extreme core allocations, in Section 5. Section 6 extends our previous results to the reverse many-to-one model in Kaneko (1976), where buyers have unitary demand, and Section 7 concludes.

2 Notations and definitions

A *transferable utility (TU) cooperative game* (N, v) is defined as a pair where N is a non-empty, finite set of *players (or agents)*, and $v : 2^N \to \mathbb{R}$ is a *coalitional function* such that $v(\emptyset) = 0$. The value $v(S)$ represents the worth of any coalition $S \subseteq N$. Since the player set N remains fixed throughout the paper, we may identify the game with its coalitional function v. The game (N, v) is said to be *superadditive* if for any two disjoint coalitions $S, T \subseteq N$ (i.e., $S \cap T = \emptyset$), it holds that $v(S \cup T) \geq v(S) + v(T)$. A coalition $R \subseteq N$ is called *inessential* in the game v if it can be written as a nontrivial partition $R = S \cup T$, with $S, T \neq \emptyset$ and $S \cap T = \emptyset$, such that $v(R) \leq v(S) + v(T)$. In superadditive games, weak majorization can occur only as an equality. A coalition is called *essential* if it is non-empty and not inessential. It is worth noting that singleton coalitions (i.e., $\{i\}$ for $i \in N$) are always essential, and any inessential coalition's value is weakly majorized by a partition composed entirely of essential coalitions.

For a given game (N, v), a *payoff allocation* is a vector $x \in \mathbb{R}^N$ assigning a payoff to each player. The total payoff to a coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. A payoff allocation x is said to be *efficient* if it satisfies $x(N) = v(N)$. The set of *imputations*, denoted by $I(v)$, includes all efficient allocations that are also *individually rational*, that is, $x_i \geq v(\{i\})$ for all $i \in N$. The *core*, denoted by $C(v)$, is the subset of imputations that are *coalitionally rational*, i.e., $x(S) \geq v(S)$ for all $S \subseteq N$. Since the coalitional rationality conditions for inessential coalitions are implied by those for essential coalitions, they can be omitted: the core and the essential-core coincide.

Given a game (N, v), the game (N, v^*) defined by $v^*(S) = v(N) - v(N \setminus S)$ for all $S \subseteq N$ is called the *dual game*. Notice that $v^*(\emptyset) = 0$ and $v^*(N) = v(N)$ for any game (N, v). It is easily seen that the core of any coalitional game coincides with the *anti-core* of its dual game, that is,

$$\mathbf{C}(v) = \mathbf{C}^*(v^*) := \{x \in \mathbb{R}^N : x(N) = v^*(N),\ x(S) \leq v^*(S)\ \forall S \subseteq N\}. \tag{1}$$

It follows that if $i \in N$ is a *null player* in game v (i.e. $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N$, in particular, $v(\{i\}) = 0$), its payoff is $x_i = 0$ in any core allocation $x \in C(v)$. Indeed, then $v(N) = v(N \setminus \{i\}) + v(\{i\}) \leq x(N \setminus \{i\}) + x_i = x(N) = v(N)$, implying both $x(N \setminus \{i\}) = v(N \setminus \{i\})$ and $x_i = v(i) = 0$.

An order on the set of players N is a bijection $\sigma : \{1, 2, \ldots, n\} \to N$, where for all $i \in \{1, 2, \ldots, n\}$, $\sigma_i = \sigma(i)$ is the player that occupies position i. For a given order σ, $P_i^\sigma = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ denotes the set of predecessors of agent $i \in N$. For each order σ on the player set N of game (N, v), a *marginal payoff vector* $m^{\sigma,v}$ is defined by $m^{\sigma,v}_{\sigma_i} = v(P^\sigma_{\sigma_i} \cup \{\sigma_i\}) - v(P^\sigma_{\sigma_i})$ for all $i \in N$. Whenever a marginal payoff vector is in the core, then it is an extreme core allocation.

Hamers et al. (2002) showed that each extreme core allocation of a one-to-one assignment game is a marginal payoff vector. Nevertheless, the opposite implication only holds in convex assignment games.

Núñez and Solymosi (2017) studied other lexicographic allocation procedures for coalitional games looking for a characterization and a computation procedure of their extreme core points. Given a game (N, v) and over the set $\mathbf{Ra}^*(N, v) = \{x \in \mathbb{R}^N : x(S) \leq v(N) - v(N \setminus S)$ for all $S \subseteq N\}$ of dual coalitionally rational payoff vectors, the following lexicographic maximization procedure is proposed: for any order σ of the players, the σ-*lemaral* vector $\overline{r}^{\sigma,v} \in \mathbb{R}^N$ is defined by, for all $i \in \{1, 2, \dots, n\}$,

$$\overline{r}^{\sigma,v}_{\sigma_i} = \max\left\{x_{\sigma_i} \ : \ x \in \mathbf{Ra}^*(N, v),\, x_{\sigma_l} = \overline{r}^{\sigma,v}_{\sigma_l}\ \forall l \in \{1, \dots, i-1\}\right\}, \tag{2}$$

which trivially leads to

$$\overline{r}^{\sigma,v}_{\sigma_i} = \min\left\{v^*(Q \cup \{\sigma_i\}) - \overline{r}^{\sigma,v}(Q) \ : \ Q \subseteq P^{\sigma}_{\sigma_i}\right\}. \tag{3}$$

Notice that in the σ-lemaral vector, the first player in the order maximizes its payoff on the set $\mathbf{Ra}^*$, the second player maximizes its payoff over those dual coalitional rational payoff vectors that allocate $\overline{r}^{\sigma,v}_{\sigma_1}$ to the first player, and so on. It is proved in (Núñez and Solymosi, 2017) that the set of extreme core allocations of a one-to-one assignment game coincides with the set of lemaral vectors.

3 The many-to-one assignment market and game

A many-to-one assignment market $\gamma = (F, W, A, r)$ consists of a finite set of firms $F = \{f_1, f_2, \dots, f_m\}$, a finite set of workers $W = \{w_1, w_2, \dots, w_n\}$ where the number of firms m can be different from the number of workers n, a valuation matrix denoted by $A = (a_{ij})_{(i,j)\in F\times W}$ where a_{ij} represents the pairwise income that is obtained when firm $i \in F$ hires worker $j \in W$. Each firm $i \in F$ would like to hire up to $r_i \geq 0$ workers and each worker $j \in W$ can work for at most one firm. Let $N = F \cup W$ be the set of all agents. We sometimes denote a generic firm and a generic worker by i and j, respectively. Then, a many-to-one assignment market is described by the quadruple $\gamma = (F, W, A, r)$.

A *matching* μ for the market $\gamma = (F, W, A, r)$ is a set of $F \times W$ pairs such that each firm $i \in F$ appears in at most r_i pairs and each worker $j \in W$ in at most one pair. We denote by $\mathcal{M}(F, W, r)$ the set of all possible matchings for market γ. A matching $\mu \in \mathcal{M}(F, W, r)$ is said to be *optimal* for γ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ holds for any other matching $\mu' \in \mathcal{M}(F, W, r)$. We denote by $\mathcal{M}_A(F, W, r)$ the set of optimal matchings for the market γ. Given a matching $\mu \in \mathcal{M}(F, W, r)$, the set of workers matched to firm $i \in F$ under μ is denoted by $\mu(i) = \{j \in W \mid (i, j) \in \mu\}$. It is convenient to represent the set of workers who remain unmatched under μ as $\mu(f_0)$, defined by $\mu(f_0) = W \setminus \bigcup_{i\in F} \mu(i)$. Note that if $i \neq k \in F$, then $\mu(i) \cap \mu(k) = \emptyset$. Therefore, the union $\mu(f_0) \cup \bigcup_{i\in F} \mu(i) = W$ forms a partition of the set of workers.

Given a many-to-one assignment market $\gamma = (F, W, A, r)$, we define the income maximization linear programming problem as follows:

$$\mathcal{V}(F,W) = \max \sum_{i\in F}\sum_{j\in W} a_{ij}x_{ij} \tag{4}$$
$$\text{s. t.} \quad \sum_{j\in W} x_{ij} \leq r_i,\ i\in F$$
$$\sum_{i\in F} x_{ij} \leq 1,\ \ j\in W$$
$$x_{ij} \geq 0, \quad (i,j)\in F\times W.$$

It is well-known that any variable in a basic feasible solution of this linear program, when the right-hand sides are integers, takes on only integer values. Therefore, due to the worker capacity constraints, any such variable x_{ij} is either 0 or 1. As a result, the relation $(i,j)\in\mu \ \leftrightarrow\ x_{ij}=1$ defines a bijection between the set of basic feasible solutions of the LP and the set of matchings $\mu\in\mathcal{M}(F,W,r)$. Hence, the optimal value of (4) corresponds to the maximum total value of all matched pairs, subject to the firms' capacity constraints.

Given a subset of firms $S\subseteq F$ and a subset of workers $T\subseteq W$, we define the corresponding submarket as $\gamma_{(S,T)} = (S,T,A_{(S,T)},r_S)$ where $A_{(S,T)}$ is the restriction of the payoff matrix A to $S\times T$, and r_S is the restriction of the capacity vector r to the set S.

Next, we associate a coalitional game with transferable utility (TU-game) to this type of two-sided matching markets. Given a many-to-one assignment market $\gamma = (F,W,A,r)$, its associated *many-to-one assignment game* is the pair (N,v_γ), where $N=F\cup W$ is the set of players and the coalitional function is defined by $v_\gamma(S\cup T) = \max\limits_{\mu\in\mathcal{M}(S,T,r_S)} \sum\limits_{(i,j)\in\mu} a_{ij}$ for all $S\subseteq F$ and $T\subseteq W$. When no confusion arises, we denote the corresponding coalitional function simply by v instead of v_γ for a given market γ.

We denote coalition $S\cup T$ with $S\subseteq F$ and $T\subseteq W$ by (S,T), in particular, one-sided coalitions by $(\emptyset,T)$ and $(S,\emptyset)$. As the union of matchings for disjoint coalitions results in a matching for the union of those coalitions, that is, if $\mu\in\mathcal{M}(S,T,r_S)$ and $\mu'\in\mathcal{M}(S',T',r_{S'})$ with $S\cap S'=\emptyset$ and $T\cap T'=\emptyset$, then $\mu\cup\mu'\in\mathcal{M}(S\cup S',T\cup T',r_{S\cup S'})$, it follows that many-to-one assignment games are superadditive. On the other hand, suppose $\nu\in\mathcal{M}(S,T,r_S)$ is an optimal matching for coalition (S,T), i.e., $v_\gamma(S,T) = \sum\limits_{(i,j)\in\nu} a_{ij} = \sum\limits_{i\in S}\sum\limits_{j\in\nu(i)} a_{ij}$. Then, since $v_\gamma(i,\nu(i)) = \sum\limits_{j\in\nu(i)} a_{ij}$ for all $i\in S$, we get $v_\gamma(S,T) = \sum\limits_{i\in S} v_\gamma(i,\nu(i))$.

Moreover, since $(S,T) = (\emptyset,\nu(f_0))\cup\bigcup\limits_{i\in S}(i,\nu(i))$, where $\nu(f_0)$ denotes the unmatched workers in T under ν, and $v_\gamma(\nu(f_0)) = 0 = \sum\limits_{j\in\nu(f_0)} v_\gamma(j)$, we arrive at the following observations.

Proposition 1. *In many-to-one assignment games, the following types of coalitions are inessential:*

- *any coalition containing at least two firms,*

- *any single-firm coalition containing more workers than the firm's capacity,*
- *any one-sided coalition containing at least two players.*

This sufficient condition for inessentiality can be summarized by saying that if a coalition contains a subcoalition that violates any of the capacity constraints, then that coalition is inessential.

Consequently, in an $(m+n)$-player many-to-one assignment game, among the $2^{m+n}-1$ non-empty coalitions, at most $\sum_{i=1}^{m}\sum_{t=1}^{r_i}\binom{n}{t} \leq 2^n - 2$ coalitions can be essential. This exponential upper bound is sharp (e.g., when all pairwise income values a_{ij} are positive, $m = 2$, and $n = r_1 + r_2$).

We will see in Proposition 2 that the core is already described by a quadratic number of easily identifiable essential coalitions.

As in any coalitional game, the main concern is how to share the worth of the grand coalition (i.e., the total income) among all agents. To do so, we focus on the solution concept known as the *core*. Unlike in one-to-one assignment games, where the only essential coalitions are individuals and matched pairs, here (see Proposition 1) unstability may arise from a group of workers and a firm who can benefit by recontracting between themselves instead of their prescribed agreements.

3.1 Core and competitive salaries

Given a many-to-one assignment market $\gamma = (F, W, A, r)$ and $\mu \in \mathcal{M}_A(F, W, r)$ an optimal matching, $(x, y) \in \mathbb{R}^F_+ \times \mathbb{R}^W_+$ is in the core $C(v_\gamma)$ of the associated game if and only if for every firm $i \in F$,

$$x_i + \sum_{j\in T} y_j \geq \sum_{j\in T} a_{ij} = v_\gamma(i, T) \text{ for all } T \subseteq W \text{ with } |T| \leq r_i (\text{with equality for } T = \mu(i)) \tag{5}$$

and the payoff to unassigned firms or workers is zero.

The above description of the core of a many-to-one assignment game is based on Proposition 1 and the general equivalence of the core and the essential-core. As we remarked there, this description is still of exponential size (in the number of players). Next, we present a quadratic-size equivalent description of the core, just in terms of the workers' payoffs. It rests on the observation that in the essential-core description (5) only those single-firm coalitions are needed for which $|T \cap \mu(i)| = r_i - 1$.

For the sake of exposition, first we balance the model, if needed. In case the total capacity of the firms $\sum_{i\in F} r_i$ exceeds the number of workers n, we introduce $\sum_{i\in F} r_i - n > 0$ dummy workers who can only generate zero income with any firm. Exclusively from this situation, in case of $n > \sum_{i\in F} r_i$, we introduce a dummy firm, say f_0, requiring at most $r_0 = n - \sum_{i\in F} r_i > 0$ number of workers, but who can only generate zero income with any worker. Technically, if needed, we extend matrix A with $\sum_{i\in F} r_i - n > 0$ full 0 columns, or with one full 0 row. This clearly means that we extend the associated many-to-one assignment game v with one or more null players. Since the core payoff to any null player j is $x_j = 0$, the core of the original game is precisely the $x_j = 0$ section of the core of the extended game, we can assume without loss of generality that the market

$\gamma = (F, W, A, r)$ is *capacity-balanced*, i.e. $n = \sum_{i\in F} r_i$ holds. To keep the exposition simple, we do not introduce any new notation for the possible extended models.

Given a capacity-balanced market $\gamma = (F, W, A, r)$ ($n = \sum_{i\in F} r_i$), we will always assume, due to the non-negativity of matrix A without loss of generality, that in any optimal matching $\mu \in \mathcal{M}_A(F, W, r)$ any firm $i \in F$ is assigned precisely r_i workers ($|\mu(i)| = r_i$), and no worker is unmatched under μ. For any worker $j \in W$, let $j^\mu \in F$ denote the unique firm for which $j \in \mu(j^\mu)$ holds, that is $j^\mu = \mu^{-1}(j)$.

The next description of the core in terms of the workers' payoffs follows easily from (5) and is a simplification of the one already given in (Sotomayor, 2002) for the general, not necessarily capacity-balanced, market.

Proposition 2. *Given a capacity-balanced many-to-one assignment market $\gamma = (F, W, A, r)$, let $\mu \in \mathcal{M}_A(F, W, r)$ be an optimal matching. Then, $(x, y) \in \mathbb{R}^F \times \mathbb{R}^W$ is in the core of the associated game $C(v_\gamma)$ if and only if*

(i) $0 \le y_j \le a_{j^\mu j}$ *for any* $j \in W$*;*

(ii) $y_k - y_j \ge a_{j^\mu k} - a_{j^\mu j}$ *for any* $j, k \in W$ *such that* $j^\mu \ne k^\mu$*;*

(iii) $x_i = \sum_{j\in\mu(i)} (a_{ij} - y_j)$ *for all* $i \in F$*.*

Notice that the number of constraints is $2n = \sum_{i\in F} 2r_i$ in item (i), $\sum_{i\in F} r_i(n - r_i) = n\sum_{i\in F} r_i - \sum_{i\in F} r_i^2 \le n^2 - \sum_{i\in F} r_i = n^2 - n$ in item (ii), and $m \le n$ in item (iii), altogether at most $n^2 + 2n$.

Also, given any vector of salaries $y \in \mathbb{R}^W$ that satisfies constraints (i) and (ii) above for some optimal matching μ, the payoff to each firm is uniquely determined. Let us denote by $C(W)$ the set of salaries (or wages) that satisfy (i) and (ii), that is, the projection of the core to the space of workers' payoffs. It is proved in (Sotomayor, 2002) that $C(W)$ is endowed with a lattice structure under the partial order induced by $\mathbb{R}^W$. In fact, the reader will see that constraints (i) and (ii) in Proposition 2 define what is named a 45-degree polytope in (Quint, 1991).

Here is an illustrative example of the above core description.

Example 3. Consider a many-to-one assignment market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3\}$ are the set of firms and the set of workers respectively, and the capacities of the firms are $r = (2, 1)$. The pairwise valuation matrix is the following:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 8 & 6 & 3 \\ 7 & 6 & 4 \end{pmatrix} \end{array}.$$

Since the (unique) optimal matching assigns workers w_1 and w_2 to firm f_1, and w_3 to firm f_2, in any core allocation $x_1 = (8 + 6) - y_1 - y_2$ and $x_2 = 4 - y_3$ hold. Henceforth, in terms of the workers payoffs, the core is described by the following system (given in two equivalent forms):

$$
\begin{array}{cccrl}
y_1 & y_2 & y_3 & \geq & 0 \\
\hline
y_1 & & & \leq & 8 \\
& y_2 & & \leq & 6 \\
& & y_3 & \leq & 4 \\
\hline
-y_1 & & y_3 & \geq & -5 = 3-8 \\
& -y_2 & y_3 & \geq & -3 = 3-6 \\
\hline
y_1 & & -y_3 & \geq & 3 = 7-4 \\
& y_2 & -y_3 & \geq & 2 = 6-4 \\
\hline
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
\hline
0 & \leq & y_1 & & & \leq & 8 \\
0 & \leq & & y_2 & & \leq & 6 \\
0 & \leq & & & y_3 & \leq & 4 \\
\hline
3 & \leq & y_1 & & -y_3 & \leq & 5 \\
2 & \leq & & y_2 & -y_3 & \leq & 3 \\
\hline
\end{array}
$$

Notice the similarities to the one-to-one assignment case, but due to the capacity $r_1 = 2$ of firm f_1, there is no direct relation between the payoffs to its optimally matched workers, two-way direct pairwise comparisions are only between workers assigned to different firms.

Figure 1 illustrates the $C(W)$ of this example, where the 45-degree lattice structure can be seen.

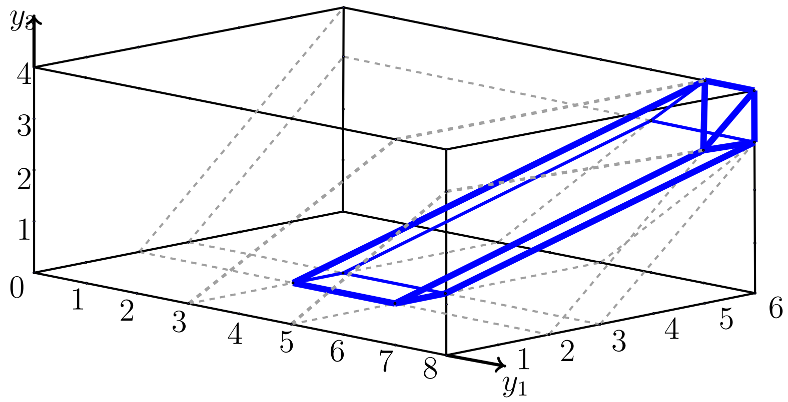

Figure 1: $C(W)$ for the many-to-one market of Example 3

From the description of the core in Proposition 2, it follows straightforwardly that it coincides with the set of competitive equilibrium payoff vectors, that is, $C(W)$ coincides with the set of competitive salary vectors. To see that, we adapt the usual definition of competitive prices to our job market setting with salaries.

Let $\gamma = (F, W, A, r)$ be a many-to-one job market, $\mu \in \mathcal{M}(F, W, r)$ a matching and $y \in \mathbb{R}^W_+$ a vector such that y_j is the salary of worker $j \in W$. *The pair* (μ, y) *is a competitive equilibrium* for this market if and only if:

1. For each $i \in F$, $\mu(i) \in D_i(y)$, where $D_i(y)$ is the set of $R \subseteq W$, $|R| \leq r_i$ such that

$$\sum_{j \in R}(a_{ij} - y_j) \geq \sum_{j \in S}(a_{ij} - y_j), \text{ for all } S \subseteq W \text{ such that } |S| \leq r_i \text{ and}$$

2. $y_j = 0$ if worker j is not assigned by μ to any firm $i \in F$.

We then say that $y \in \mathbb{R}^W$ is a vector of competitive salaries and it is compatible with matching μ. It is well known that μ must be an optimal matching for γ and that the competitive salary vector y is also compatible with any other optimal matching.

4 The set of extreme competitive salary vectors

Generically, the core of a many-to-one assignment game contains infinitely many allocations, each of them supported by a vector of competitive salaries. In particular, this is the case when the optimal matching is unique, and then the dimension of the core is $(m+n) - m = n$.

Special attention has been paid to the vectors of maximum and minimum competitive prices (or salaries in our case). For many-to-one assignment market $\gamma = (F, W, A, r)$, as a particular case of the model in (Gul and Stachetti, 1999), and normalizing at zero the reservation values of the workers, these two extreme vectors of competitive salaries can be obtained from

$$\overline{y}_j = v_\gamma(F \cup W) - v_\gamma(F \cup (W \setminus \{j\}), \text{ for all } j \in W, \quad (6)$$

$$\underline{y}_j = v_{\gamma^{+j}}(F \cup (W \cup \{j'\})) - v_\gamma(F \cup W), \text{ for all } j \in W, \quad (7)$$

where the value $v_{\gamma^{+j}}(F \cup (W \cup \{j'\}))$ is obtained by duplicating in the valuation matrix A the column of worker j and looking for an optimal matching among those that do not allocate the two copies of worker j to the same firm.

It is well known that if all firms have a unitary capacity, then $\underline{y}_j = v_\gamma((F \setminus \{j^\mu\}) \cup W) - v_\gamma((F \setminus \{j^\mu\}) \cup (W \setminus \{j\}))$ and the maximum core payoff of the firm assigned to j is its marginal contribution $\overline{x}_{j^\mu} = v_\gamma(F \cup W) - v_\gamma((F \setminus \{j^\mu\}) \cup W)$. In the many-to-one case, those firms with capacity greater than one may not attain their marginal contribution in the core. Take for instance firm f_1 in Example 3, since the minimum competitive salaries are $\underline{y} = (3, 2, 0)$, the maximum core payoff of this firm is $(8-3) + (6-2) = 9$ that is below $v_\gamma(F \cup W) - v_\gamma((F \setminus \{f_1\}) \cup W) = 11$.

Besides the vectors of maximum and minimum competitive salaries, there may be several other extreme points in the set of competitive salaries, which correspond to the set of extreme core allocations. The description of these extreme points gives information about how large this set is, and how many different stable agreements can be attained in the market. The digraph we introduce next, associated with each vector of competitive salaries, provides a characterization of all extreme vectors of competitive salaries, not just of the maximum and minimum ones.

Definition 4. Let $\gamma = (F, W, A, r)$ be a capacity-balanced many-to-one assignment market and $\mu \in \mathcal{M}_A(F, W, r)$ be an optimal matching. For each vector of competitive salaries $y \in C(W)$, we define the *tight digraph* (W_0, E^y) with set of nodes $W_0 := W \cup \{0\}$, where 0 is a fictitious worker whose salary is fixed to $y_0 = 0$ and with the set of arcs E^y

such that

$$(j,k) \in E^y \quad \leftrightarrow \quad y_k - y_j = \begin{cases} 0 & \text{if } j = 0, k \in W; \\ -a_{j^\mu j} & \text{if } j \in W, k = 0; \\ a_{j^\mu k} - a_{j^\mu j} & \text{if } j \in W, k \in W \setminus \mu(j^\mu). \end{cases}$$

This tight digraph is inspired by the one introduced in (Balinski and Gale, 1990) and also used in (Hamers et al., 2002) to study extreme core points of the one-to-one assignment game. There, the nodes of the graph consist of the agents on both sides of the market, not just from one side as we do for the many-to-one case. And also, their graph is not directed since it is based on the constraints $x_i + y_j \geq a_{ij}$ where both variables have the same sign. They find that the extreme core points are those core points with a tight graph that has an agent with zero payoff in each component. In our setting, we replace that property with connectedness to the fixed 0 payoff node, hence connectedness of the underlying undirected graph (the base-graph) of the tight digraph. Besides, we also characterize the maximum and the minimum competitive salary vectors with an (easily verifiable) additional property of the tight digraph.

Before the general discussion, we illustrate the idea and foreshadow the results on the market situation of Example 3.

Example 5. We revisit Example 3 and introduce a fictitious worker 0 who is optimally matched to a fictitious firm with capacity 1, denoted 0^μ, because in any matching of the extended capacity-balanced market the two fictitious agents are required to be paired. Their payoffs are fixed to 0. The virtual possibility of being matched to the fictitious agent on the other side will represent the outside option of an agent, thus the pairwise surplusses with them are set to zero. The extended market, with the optimally assigned firm-worker pairs boxed, is given on the left below. For brevity, we represent the workers and the firms by their index. On the right below we present the description of $C(W)$ where all constraints are written in a unified way.

	0	1	2	3	
0^μ	$\boxed{0}$	0	0	0	$r_{0^\mu} = 1$
1	0	$\boxed{8}$	$\boxed{6}$	3	$r_1 = 2$
2	0	7	6	$\boxed{4}$	$r_2 = 1$

-0	$+y_1$			$\geq$	$0 = 0 - 0$
-0		$+y_2$		$\geq$	$0 = 0 - 0$
-0			$+y_3$	$\geq$	$0 = 0 - 0$
$+0$	$-y_1$			$\geq$	$-8 = 0 - 8$
$+0$		$-y_2$		$\geq$	$-6 = 0 - 6$
$+0$			$-y_3$	$\geq$	$-4 = 0 - 4$
	$-y_1$		$+y_3$	$\geq$	$-5 = 3 - 8$
		$-y_2$	$+y_3$	$\geq$	$-3 = 3 - 6$
	$+y_1$		$-y_3$	$\geq$	$3 = 7 - 4$
		$+y_2$	$-y_3$	$\geq$	$2 = 6 - 4$

Due to this special structure, we associate a directed graph that represents by arcs the inequalities which are tight (satisfied as equality) at a given $y \in C(W)$ and decide if y is an extreme point by checking whether the base-graph is connected.

Recall that in this market the minimum competitive salary vector is $(3, 2, 0)$ that makes the following inequalities tight: $-0 + y_3 = 0$, $y_1 - y_3 = 3$, and $y_2 - y_3 = 2$. The associated digraph is pictured on the left in Figure 2. At the maximum competitive salary vector $(8, 6, 4)$ the following inequalities are tight: $0 - y_1 = -8$, $0 - y_2 = -6$, $0 - y_3 = -4$, and $y_2 - y_3 = 2$. The associated digraph is pictured on the right in Figure 2.

Figure 2: digraph of minimum vector $(3, 2, 0)$, digraph of maximum vector $(8, 6, 4)$

In both cases the base-graph is connected. Notice that in the tight digraph of the minimum competitive salary vector $(3, 2, 0)$ node 0 is the only source, while in the tight digraph of the maximum competitive salary vector $(8, 6, 4)$ node 0 is the only sink.

Similarly, the tight digraphs associated with CE vectors $(3, 3, 0)$ and $(7, 6, 4)$, pictured, respectively, on the left and on the right in Figure 3, are both connected. Thus, both vectors are also extreme points of $C(W)$. However, in neither of these tight digraphs node 0 is the only source or the only sink.

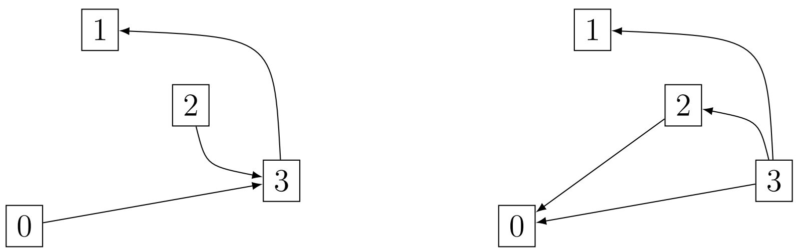

Figure 3: digraph of extreme vector $(3, 3, 0)$, digraph of extreme vector $(7, 6, 4)$

Indeed, in case of $(3, 3, 0)$, node 2 is also a source, indicating that none of the constraints which contains $-y_2$ is tight, hence y_2 can be decreased (with a sufficiently small positive amount) without leaving the feasible solution set. Therefore, $(3, 3, 0)$ cannot be the minimum competitive salary vector. Similarly, in case of $(7, 6, 4)$, node 1 is also a sink, indicating that none of the constraints which contain $+y_1$ is tight, hence y_1 can be increased (with a sufficiently small positive amount) without violating any of the lower-bound constraints. Therefore, $(7, 6, 4)$ cannot be the maximum competitive salary vector.

We now formally establish the characterization of the extreme vectors of competitive salaries of the many-to-one assignment game by properties of the corresponding tight digraphs (Definition 4).

Theorem 6. *Let $\gamma = (F, W, A, r)$ be a capacity-balanced many-to-one assignment market in which μ is an optimal matching. Then*

(A) *$y \in C(W)$ is an extreme vector of competitive salaries if and only if the base-graph of the associated tight digraph (W_0, E^y) is connected.*

(B) *$y \in C(W)$ is the minimum vector of competitive salaries if and only if its tight digraph (W_0, E^y) contains a 0-sourced directed spanning tree (i.e. all arcs of the spanning tree are directed away from node 0).*

(C) *$y \in C(W)$ is the maximum vector of competitive salaries if and only if its tight digraph (W_0, E^y) contains a 0-sinked directed spanning tree (i.e. all arcs of the spanning tree are directed towards node 0).*

Proof. First, we prove the "*only if*" part of characterization (A). Suppose on the contrary that the base-graph of the tight digraph associated with an extreme vector $y \in C(W)$ is not connected. Then let $W' \subset W$ be the node set of a component which does not contain node 0. Now, let us define $\varepsilon = \min\{(y_k - y_j) - (a_{j^\mu k} - a_{j^\mu j}) : j \in W', k \in W_0 \backslash W'\}$, with $y_k = a_{j^\mu k} = 0$ if $k = 0$. Since there are no arcs between W' and the rest of the nodes $W_0 \setminus W'$, we have $\varepsilon > 0$. If we define $y'_j = y_j + \varepsilon$, $y''_j = y_j - \varepsilon$ for all $j \in W'$, and $y'_k = y''_k = y_k$ for all $k \in W_0 \setminus W'$, both vectors y' and y'' also belong to $C(W)$. However, $y = \frac{1}{2}y' + \frac{1}{2}y'' \in C(W)$, which contradicts the assumption that y is an extreme point.

Second, we prove the "*if*" part of (A). If the base-graph of the tight digraph associated with a vector $y \in C(W)$ is connected, then there is a path from node 0 to any node $j \in W$, that is a sequence of nodes $0 = j_0, j_1, \ldots, j_k = j$ with $k \geq 1$ such that any two consecutive nodes are the two endpoints of an arc. If $(j_h j_{h+1}) \in E^\mu$ $(0 \leq h \leq k-1)$ then it is called a forward arc, if $(j_{h+1} j_h) \in E^\mu$ $(0 \leq h \leq k-1)$ then it is called a backward arc. In case nodes j_h and j_{h+1} $(0 \leq h \leq k-1)$ are connected by both types of arcs in the tight digraph, we choose one of them arbitrarily for the path. If we add the equations $y_{h+1} - y_h = a_{h^\mu h+1} - a_{h^\mu h}$ related to the forward arcs in this path, and subtract the sum of the equations $y_h - y_{h+1} = a_{(h+1)^\mu h} - a_{(h+1)^\mu h+1}$ related to the backward arcs, all variables y_h, $1 \leq h \leq k-1$ (if any) cancel out, only $y_j - y_0 = y_j$ remains on the left side. Thus, we get $y_j = \sum_{(h,h+1) \in E^\mu} (a_{h^\mu h+1} - a_{h^\mu h}) - \sum_{(h+1,h) \in E^\mu} (a_{(h+1)^\mu h} - a_{(h+1)^\mu h+1})$. Since all salaries y_j $(j \in W)$ are uniquely determined by the tight constraints, their vector y is an extreme point of $C(W)$.

We only prove the characterization for the minimum vector of competitive salaries in (B), the proof for the maximum vector in (C) goes in an analogous way.

Assume first that $y \in C(W)$ is the minimum vector of competitive salaries. If there is a node $j \in W$ with no incoming arc then all constraints in which y_j appears with $+1$ coefficient are satisfied as strict inequalities, so while keeping all other variables fixed, we can decrease y_j with a sufficiently small positive amount without violating any of these constraints. Besides, we actually increase the left hand side of those greater-or-equal inequalities in which y_j appears with -1 coefficient, and we do not change the left hand side of the rest of the constraints. This would contradict the minimality of y_j. Thus, none of the nodes $j \in W$ can be a source at the minimum vector of competitive salaries. A similar argument shows that there must be an arc going out from node 0, for otherwise we could decrease all salaries y_j ($j \in W$) with a sufficiently small positive amount without violating any CE constraint, again a contradiction to the minimality of vector y. Combining these two observations with the finiteness of the number of nodes, we conclude that there must exist a directed path (i.e. containing only forward arcs) from node 0 to any node $j \in W$, implying that the tight digraph (W_0, E^y) must contain a 0-sourced directed spanning tree.

To see the converse implication in (B) for the minimum vector of competitive salaries, assume that for an arbitrary $y \in C(W)$ the associated tight digraph (W_0, E^y) contains a 0-sourced directed spanning tree. Then there exits a directed path $0 = j_0, j_1, \ldots, j_k = j$ with $k \geq 1$ from node 0 to any node $j \in W$ containing only forward arcs. Thus, if we add the related inequalities $y'_{h+1} - y'_h \geq a_{h^\mu h+1} - a_{h^\mu h}$ along this directed path, we get $y'_j \geq \sum_{(h,h+1)\in E^\mu}(a_{h^\mu h+1} - a_{h^\mu h})$ for any feasible vector $y' \in C(W)$. For the selected $y \in C(W)$, all these constraints hold as equalities, thus, $y_j = \min\{y'_j : y' \in C(W)\}$ for all $j \in W$, implying that $y \in C(W)$ is the minimum vector of competitive salaries. □

We remark that a tight digraph might contain a directed cycle, which might even contain node 0, but only if there are alternative optimal matchings in the many-to-one assignment market. If the optimal matching is unique, like in Example 5, node 0 is either a source or a sink (but not both) in the tight digraph associated with any extreme salary vector.

We conclude this section with an immediate consequence of Theorem 6.

Corollary 7. Let $\gamma = (F, W, A, r)$ be a capacity-balanced many-to-one assignment market in which μ is an optimal matching, and let $y \in C(W)$. If y is the vector of minimum (resp. maximum) competitive salaries, then there is a worker $j \in W$ with salary $y_j = 0$ (resp. $y_j = a_{j^\mu j}$).

5 The max-min salary vectors

In this section we intend to compute the set of extreme core allocations or, equivalently, extreme competitive salary vectors of the many-to-one assignment markets. A natural first approach is to consider the relationship between the extreme core allocations and some lexicographic allocation procedures. This approach has been applied by (i) Hamers et al. (2002) to show that each extreme core allocation of a one-to-one assignment game is a marginal payoff vector and by (ii) Izquierdo et al. (2007) to see that each such extreme point is the result of a lexicographic minimization procedure on the set of rational allocations: for each order on the set of agents, let the payoff to the first player in the order be zero and, for each following agent, compute the minimum payoff that satisfies all core inequalities with his/her predecessors while preserving the payoffs that they have already been allocated. More recently, (iii), Núñez and Solymosi (2017) prove that each extreme core allocation of the one-to-one assignment game is the result of a lexicographic maximization over the set of dual rational allocations (lemarals). However, it is easy to find examples (see Example A in the Appendix) showing that none of these three procedures allows to describe all the extreme core allocations of many-to-one assignment markets.

The characterization of the extreme competitive salary vectors of the many-to-one assignment game by means of the tight digraphs given in Theorem 6 will allow to describe a procedure to obtain all these extreme points. We will see that the extreme competitive salary vectors of these games also correspond to a sequence of lexicographic optimization, where for each given order, some workers maximize their salary while some other workers minimize it, always preserving what has been allocated to their predecessors.

There are two main differences between the following definition of the max-min salary vectors and the lexicographic procedures applied to the one-to-one assignment game: only workers are now considered and each order on the set of workers must be completed with an indication of whether the worker in this position maximizes or minimizes his/her salary.

Let $\theta : \{1, \ldots, n = |W|\} \longrightarrow W$ be an order on the set of workers, where $\theta(i)$ is the worker in the ith-position, and we can also write $\theta = (j_1, j_2, \ldots, j_n)$. We denote by Σ_W the set of all orders on W. Given a worker $j \in W$, $P_j^\theta = \{k \in W \mid \theta^{-1}(k) < \theta^{-1}(j)\}$ is the set of predecessors of j according the order θ.

Then, an extension of the order θ is

$$\begin{array}{rcl} \tilde{\theta} : \{1, \ldots, n = |W|\} & \longrightarrow & W \times \{\min, \max\} \\ i & \mapsto & \tilde{\theta}(i) = \left\{ \begin{array}{l} (\theta(i), \min) = \underline{\theta}(i) \\ \text{or} \\ (\theta(i), \max) = \overline{\theta}(i), \end{array} \right. \end{array}$$

where $\underline{\theta}(i)$ means that worker is in ith position and will minimize his/her salary under some constraints. Similarly, $\overline{\theta}(i)$ means that the ith player in the order will maximize his/her salary under some constraints. We denote by $\widetilde{\Sigma}_W$ the set of all extended orders on W. Clearly, $|\Sigma_W| = n!$ and $|\widetilde{\Sigma}_W| = n! \cdot 2^n$, where $n = |W|$ is the number of workers.

Definition 8. Let (F, W, A, r) be a capacity-balanced many-to-one assignment game, μ an optimal matching, $\theta = (j_1, j_2, \ldots, j_n)$ an order on W and $\tilde{\theta}$ an extension of θ. The related *max-min salary vector* $y^{\tilde{\theta}}$ satisfies

$$y_{j_1}^{\tilde{\theta}} = \left\{ \begin{array}{lll} 0 & \text{if} & \tilde{\theta}(1) = \underline{\theta}(1) \\ a_{j_1^\mu j_1} & \text{if} & \tilde{\theta}(1) = \overline{\theta}(1), \end{array} \right.$$

and for all $1 < r \leq n$,

$$y_{j_r}^{\tilde{\theta}} = \left\{ \begin{array}{lll} \max_{j \in P_{j_r}^\theta, j^\mu \neq j_r^\mu} \{y_j - a_{j^\mu j} + a_{j^\mu j_r}, 0\} & \text{if} & \tilde{\theta}(r) = \underline{\theta}(r) \\ \min_{j \in P_{j_r}^\theta, j^\mu \neq j_r^\mu} \{y_j - a_{j_r^\mu j} + a_{j_r^\mu j_r}, a_{j_r^\mu j_r}\} & \text{if} & \tilde{\theta}(r) = \overline{\theta}(r). \end{array} \right.$$

To give an interpretation to these vectors, recall from Proposition 2 that the core constraints worker j_r must satisfy are $0 \leq y_{j_r} \leq a_{j_r^\mu j_r}$ and

$$a_{j^\mu j_r} - a_{j^\mu j} \leq y_{j_r} - y_j \leq a_{j_r^\mu j_r} - a_{j_r^\mu j}, \text{ for all } j \in W,\ j^\mu \neq j_r^\mu.$$

Then, when we reach worker j_r following order θ, the max-min vector procedure only considers the core constraints with variables from $P_{j_r}^\theta \cup \{j_r\}$ and determines a payoff (salary) for j_r that satisfies (in a tight way) either one lower core bound or one upper core bound, depending on whether the extended order $\tilde{\theta}$ determines j_r is a maximizer or a minimizer. Since all y_j values for $j \in P^{\theta}_{j_r}$ have already been set, finding y_{j_r} amounts to the elementary optimization problems given in the above definition. It is not surprising that a max-min salary vector may not be in $C(W)$, since one half of the core constraints are not checked during the procedure that builds such vector. However, we show next that if a max-min salary vector is competitive, then it is an extreme competitive vector. This same property (the fact that when they are in the core, they are extreme core points) is satisfied by the marginal worth vectors in arbitrary coalitional games and by the max-payoff vectors (Izquierdo et al., 2007) in one-to-one assignment games, which are also collections of vectors that are defined for each possible order on a player set.

Proposition 9. *Let $\gamma = (F, W, A, r)$ be a capacity-balanced many-to-one assignment market, μ an optimal matching, θ an order on W and $\tilde{\theta}$ an extension of θ. If $y^{\tilde{\theta}} \in C(W)$, then $y^{\tilde{\theta}} \in Ext(C(W))$.*

Proof. Let $y^{\tilde{\theta}} \in C(W)$. By definition of the max-min vectors, at each step of the procedure one core constraint is tight at $y^{\tilde{\theta}}$. Moreover, these equations are linearly independent since each of them involves a new worker whose salary does not take part in the previous equations. Since the membership in $C(W)$ is guaranteed by the assumption, the fact that n linearly independent constraints are tight at $y^{\tilde{\theta}}$ implies that this is an extreme point of $C(W)$. □

Now the question is whether all extreme points of $C(W)$ in a many-to-one assignment market are of this type, that is, all are max-min salary vectors related to some extended order on the set of workers. Let us consider again the market of Example 3.

Example 10. Consider again the many-to-one assignment market $\gamma = (F, W, A, r)$ with set of firms $F = \{f_1, f_2\}$ with capacities $r = (2, 1)$, set of workers $W = \{w_1, w_2, w_3\}$ with unitary capacity and pairwise valuation matrix

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \left(\begin{array}{ccc} 8 & 6 & 3 \\ 7 & 6 & 4 \end{array}\right) \end{array}.$$

We can obtain the extreme core points from the picture of the salary-core in Figure 1 and then check that all core vertices are supported by max-min salary vectors. Another approach is to compute for each of the $3! \cdot 2^3 = 48$ extended orders the associated max-min salary vectors and check their core membership. The result of this tedious but computationally straightforward excercise is given in Appendix B. It shows that all 9 extreme core vectors in this market are supported by max-min salary vectors, they are obtained from 28 extended orders, while the remaining 20 extended orders determine max-min salary vectors outside the core.

In the following table we indicate for each core vertex all extended orders such that the related max-min salary vector supports that vertex.

x_1	x_2	y_1	y_2	y_3	extended order
0	0	8	6	4	$(\overline{1},\overline{2},\overline{3})$ in any permutation, $(\overline{1},\overline{3},\underline{2}),(\overline{3},\overline{1},\underline{2}),(\overline{3},\underline{2},\overline{1})$
0	1	8	6	3	$(\overline{1},\overline{2},\underline{3}),(\overline{2},\overline{1},\underline{3}),(\overline{1},\underline{3},\overline{2}),(\overline{2},\underline{3},\overline{1})$
1	1	8	5	3	$(\overline{1},\underline{3},\underline{2})$
1	0	7	6	4	$(\overline{2},\overline{3},\underline{1}),(\overline{3},\overline{2},\underline{1}),(\overline{3},\underline{1},\overline{2}),(\overline{3},\underline{1},\underline{2}),(\overline{3},\underline{2},\underline{1})$
2	1	6	6	3	$(\overline{2},\underline{3},\underline{1})$
9	4	3	2	0	$(\underline{3},\underline{2},\underline{1}),(\underline{3},\underline{1},\underline{2})$
7	4	5	2	0	$(\underline{3},\underline{2},\overline{1}),(\underline{3},\overline{1},\underline{2})$
6	4	5	3	0	$(\underline{3},\overline{2},\overline{1}),(\underline{3},\overline{1},\overline{2})$
8	4	3	3	0	$(\underline{3},\overline{2},\underline{1}),(\underline{3},\underline{1},\overline{2})$

Characterization (A) in Theorem 6 offers explanations not just for the various multiplicity a given extreme core vector appears as max-min salary vector, but, more importantly, why the full enumeration of max-min salary vectors will always provide all extreme core vectors (see Theorem 11 below).

Take for instance extreme competitive salary vector $(7,6,4)$ and consider its tight graph drawn in Example 5. It has 4 arcs on 4 nodes, so its connected base-graph admits multiple spanning trees. We see that any of these spanning trees contains arc $(3,1)$ and two of the other three arcs (which form a cycle). For instance, the spanning tree with arcs $(2,0)$, $(3,2)$, $(3,1)$ allows only the order $(2,3,1)$ and make arcs $(2,0)$ and $(3,2)$ backward arcs and arc $(3,1)$ a forward arc. If, starting from node 0, we reach a node with a backward (resp. forward) arc, we set to maximize (resp. minimize) the payoff for that worker. Thus, in this case we get the extended order $(\overline{2},\overline{3},\underline{1})$. The related max-min salary vector is computed as follows:

$$\begin{aligned} y_2 &= a_{12} = 6, \\ y_3 &= \min\{y_2 - a_{22} + a_{23}, a_{23}\} = \min\{4,4\} = 4, \\ y_1 &= \max\{y_3 - a_{23} + a_{21}, 0\} = \max\{7,0\} = 7. \end{aligned}$$

On the other hand, the spanning tree with backward arc $(3,0)$ and forward arcs $(3,2)$, $(3,1)$ allows two extended orders compatible with the partial order induced by this 0-rooted spanning tree, namely $(\overline{3},\underline{2},\underline{1})$ and $(\overline{3},\underline{1},\underline{2})$.

This example also shows that not all max-min salary vectors belong to the core, and hence they may not lead to an extreme core allocation. Take for instance the extended order $\tilde{\theta} = (\underline{1},\overline{2},\underline{3})$. Then,

$$\begin{aligned} y_1 &= 0, \\ y_2 &= a_{12} = 6, \\ y_3 &= \max\{y_1 - a_{11} + a_{13}, y_2 - a_{12} + a_{13}, 0\} = \max\{-5,3,0\} = 3. \end{aligned}$$

The related max-min salary vector is $y^{\tilde{\theta}} = (0,6,3)$ and it does not lead to a core payoff since the constraint $y_3 - y_1 \leq a_{23} - a_{21}$, which was ignored when y_3 was minimized, is not satisfied.

Next theorem shows that although a max-min salary vector may not be an extreme core allocation, the converse inclusion always holds. As the example above illustrates, every extreme core point is supported by a max-min salary vector related to one extended order, or maybe to several of them.

Theorem 11. *Let $\gamma = (F, W, A, r)$ be a capacity-balanced many-to-one assignment market and μ an optimal matching. Then,*

$$Ext(C(W)) \subseteq \{y^{\tilde{\theta}}\}_{\tilde{\theta} \in \widetilde{\Sigma}_W}.$$

Proof. Let $y \in Ext(C(W))$ and consider the related tight digraph (W_0, E^y). From Theorem 6 (A), the base-graph is connected, hence there exists $j_1 \in W$ such that at least one of $(0, j_1) \in E^y$, meaning $y_{j_1} = 0$, or $(j_1, 0) \in E^y$, meaning $y_{j_1} = a_{j_1^\mu j_1}$, holds. If both relations hold, we pick one of them. In the first case define $\tilde{\theta}(1) = \underline{\theta}(1) = j_1$ and in the second case $\tilde{\theta}(1) = \overline{\theta}(1) = j_1$. Notice that in both cases $y^{\tilde{\theta}}_{j_1} = y_{j_1}$.

For $1 < r \leq n-1$, assume by induction hypothesis that there exists $\tilde{\theta} \in \widetilde{\Sigma}_W$ with $\tilde{\theta}(k) = j_k$ for all $1 \leq k \leq r$ such that $y^{\tilde{\theta}}_{\theta(k)} = y^{\tilde{\theta}}_{j_k} = y_{j_k}$, and show this also holds for $r+1$.

Case 1: There exists some $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ and some $j_k \in \{j_1, j_2, \ldots, j_r\}$ such that $(j_k, j) \in E^y$.

In this case, $y_j - y_{j_k} = a_{j_k^\mu j} - a_{j_k^\mu j_k}$, which implies $y_j = y_{j_k} + a_{j_k^\mu j} - a_{j_k^\mu j_k}$. Then, set $\tilde{\theta}(r+1) = \underline{\theta}(r+1) = j$, that is, $j_{r+1} = j$, and notice that, since y is a vector of competitive salaries, the inequalities $y_j \geq 0$ and $y_j \geq y_{j_h} + a_{j_h^\mu j} - a_{j_h^\mu j_h}$ hold for all $j_h \in \{j_1, \ldots, j_r\}$ with $j_h^\mu \neq j^\mu$. This guarantees that $y_{j_{r+1}} = y^{\tilde{\theta}}_{j_{r+1}}$.

Case 2: There exists some $j \in W \setminus \{j_1, j_2, \ldots, j_r\}$ and some $j_k \in \{j_1, j_2, \ldots, j_r\}$ such that $(j, j_k) \in E^y$.

In this case, $y_j - y_{j_k} = a_{j^\mu j} - a_{j^\mu j_k}$, which implies $y_j = y_{j_k} + a_{j^\mu j} - a_{j^\mu j_k}$. Then, set $\tilde{\theta}(r+1) = \overline{\theta}(r+1) = j$, that is, $j_{r+1} = j$, and notice that, since y is a vector of competitive salaries , the inequalities $y_j \leq a_{j^\mu j}$ and $y_j \leq y_{j_h} + a_{j^\mu j} - a_{j^\mu j_h}$, for all $j_h \in \{j_1, \ldots, j_r\}$ with $j_h^\mu \neq j^\mu$, hold. This shows that $y_{j_{r+1}} = y^{\tilde{\theta}}_{j_{r+1}}$.

By connectedness of the base-graph at least one of the above two cases holds, if both hold, we pick one of them, and continue building the spanning tree till all nodes in W are reached. An extended order is constructed such that the associated min-max vector coincides with the extreme core vector y, and our inductive proof ends. □

A consequence of the above theorem is that each extreme core point of a many-to-one assignment game is the result of a (computationally very simple) lexicographic optimization procedure carried out by the workers over the core. This somehow resembles the one-to-one assignment game, where each extreme core point can be obtained from a lexicographic maximization or also from a lexicographic minimization over the core. But, in both cases, all agents, firms and workers, take part in the optimization procedure.

In particular, given a market (F, W, A, r), if we take any order θ on W and consider the extended order $\tilde{\theta} = (\underline{\theta}(1), \underline{\theta}(2), \ldots, \underline{\theta}(n))$, the related max-min salary vector $y^{\tilde{\theta}}$

satisfies $y^{\tilde{\theta}}_{\theta(k)} \leq y_{\theta(k)}$ for all $(x,y) \in C(v_\gamma)$. This is because each worker's payoff at $y^{\tilde{\theta}}$ tightly satisfies some lower core bound, given the payoff of his/her predecessors. As a consequence, whenever $y^{\tilde{\theta}}$ belongs to the core, it is the worst core allocation for workers, and hence the vector of minimum competitive salaries, that supports the firm-optimal core allocation. Similarly, the worker-optimal core allocation follows from some $y^{\tilde{\theta}}$ where $\tilde{\theta} = (\overline{\theta}(1), \overline{\theta}(2), \dots, \overline{\theta}(n))$.

6 Kaneko's many-to-one buyer-seller market

The first many-to-one assignment game in the literature appears in (Kaneko, 1976), as a market between buyers and sellers where each buyer demands only one unit while each seller may have several units on sale, even from different types. If we assume for simplicity that the goods owned by a seller are of the same type, Kaneko's many-to-one assignment game is analogous to our job market assignment game from the perspective of the core and the theory of coalitional games.

Let B and S be the finite and disjoint sets of buyers and sellers respectively, $A = (a_{ij})_{(i,j)\in B\times S}$ the pairwise valuation matrix and $r = (r_j)_{j\in S}$ the capacities of the sellers. Assume the market is capacity-balanced, that is $\sum_{j\in S} r_j = |B|$. By projecting the core of this game to the payoffs of the buyers (which is now the side with unitary capacity agents) analogously to Proposition 2 we obtain that $(x,y) \in \mathbb{R}^B \times \mathbb{R}^S$ is in the core of the associated game $C(v_\gamma)$, where $\gamma = (B, S, A, r)$, if and only if, for any optimal matching μ,

(i) $0 \leq x_i \leq a_{i\mu(i)}$ for any $i \in B$;

(ii) $x_k - x_i \geq a_{k\mu(i)} - a_{i\mu(i)}$ for any $i, k \in B$ such that $\mu(k) \neq \mu(i)$;

(iii) $y_j = \sum_{i\in j^\mu} (a_{ij} - x_i)$ for all $j \in S$.

From this description of the core of Kaneko's assignment market, that we may call the buyers core $C(B)$, it follows the possibility of defining the tight digraph associated with each core element. Now this graph at $x \in C(B)$ will have set of nodes B and directed arcs related to those core inequalities that are tight at x, in a way analogous to Definition 4. As a consequence, we obtain a characterization of the extreme core allocations by means of the connectedness of its base-graph, and characterizations of the buyers-optimal core element and the sellers-optimal core element parallel to those in Theorem 6: $x \in C(B)$ is the minimum core payoff vector for buyers if and only if its tight digraph contains a 0-sourced directed spanning tree, and it is the maximum core payoff vector for buyers if its tight digraph contains a 0-sinked directed spanning tree.

Also, a set of max-min payoff vectors $\{x^{\tilde{\theta}}\}_{\tilde{\theta}\in\tilde{\Sigma}}$ can be defined, one for each extended order on the set of buyers, and each extreme element of $C(B)$ is proved to be of this type, in a result parallel to Theorem 11.

However, regarding the set of competitive equilibrium payoff vectors, the two models clearly differ. Kaneko (1976) already shows by means of an example that although every competitive equilibrium payoff vector is in the core, not all core elements are supported by competitive prices. This is quite straightforward since in the above core description, two units from the same seller $j \in S$ that are sold to two different buyers $i, k \in B$ may have different price: $a_{ij} - x_i$ and $a_{kj} - x_k$. It is easy to see that the subset of core elements where the units of each seller are sold at the same price is the set of competitive equilibria payoff vectors.

Proposition 12. *Let $\gamma = (B, S, A, r)$ be a capacity-balanced many-to-one assignment market where buyers have unitary capacity and μ an optimal matching. Then, $(x, y) \in \mathbb{R}^B \times \mathbb{R}^S$ is a competitive equilibrium payoff vector if and only if*

(i) $0 \le x_i \le a_{i\mu(i)}$ *for any* $i \in B$;

(ii) $x_k - x_i \ge a_{k\mu(i)} - a_{i\mu(i)}$ *for any* $i, k \in B$;

(iii) $y_j = \sum_{i \in j^\mu} (a_{ij} - x_i)$ *for all* $j \in S$.

Notice that the difference with the core, and with the CE equilibrium payoffs of our initial many-to-one job market, lies in the fact that inequality (ii) is required for each pair of buyers, not just for those that are not optimally matched to the same seller. This implies that if $i, k \in B$ are such that $\mu(i) = \mu(k) = j$, then (ii) gives $x_k - x_i = a_{k\mu(k)} - a_{i\mu(i)}$ which means that both units are sold at the same price: $p_j = a_{k\mu(k)} - x_k = a_{i\mu(i)} - x_i$.

Example 13. Consider the market $\gamma = (B, S, A, r)$ where the set of buyers is $B = \{b_1, b_2, b_3\}$, the set of sellers is $S = \{s_1, s_2\}$, the capacities of the sellers are $r = (2, 1)$ and the valuation matrix is

$$A = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{pmatrix} 8 & 7 \\ 6 & 6 \\ 3 & 4 \end{pmatrix} \end{array}.$$

There is only one optimal matching $\mu = \{(b_1, s_1), (b_2, s_1), (b_3, s_3)\}$ and the core of this market consists of the set of payoff vectors $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^2$ such that

$$\begin{array}{lll} 0 \le x_1 \le 8 & 3 \le x_1 - x_3 \le 5 & y_1 = (8 - x_1) + (6 - x_2) \\ 0 \le x_2 \le 6 & 2 \le x_2 - x_3 \le 3 & y_2 = 4 - x_3 \\ 0 \le x_3 \le 4 & & \end{array} \tag{8}$$

Notice that the valuation matrix is the transposed of Example 3, and the capacities of sellers coincide with those of firms in that initial example. As a consequence notice that $C(B)$ coincides with $C(W)$ there. Hence, in our buyer-seller market, $(\overline{x}, \underline{y}) = (8, 6, 4; 0, 0)$ is the best core allocation for buyers while $(\underline{x}, \overline{y}) = (3, 2, 0; 9, 4)$ is the best core allocation for sellers. However, in $(\underline{x}, \overline{y})$, s_1 sells one unit to b_1 at the price $p_{1b_1} = a_{11} - x_1 = 5$ and sells a second unit to b_2 at the price $p_{1b_2} = a_{21} - x_2 = 4$, which means that $(3, 2, 0; 9, 4)$ is not supported by a competitive equilibrium.

To obtain the set of CE payoff vectors of this example, $CE(B)$, we only need to add to the set of inequalities (8) the fact that the two units of s_1 are sold at the same price,

$8-x_1=6-x_2$, that is $x_1-x_2=2$. By representing $CE(B)$, it is easy to check that it is the polytope spanned by the following four extreme vectors: $(4,2,0)$, $(5,3,0)$, $(8,6,3)$, and $(8,6,4)$. Then, the minimum CE payoff vector for the buyers is $(4,2,0)$, related to the CE prices $p_1=p_2=4$.

Notice that the maximum payoff of the buyers in the core, $(8,6,4)$ satisfies the additional equation $x_1-x_2=2$ and hence it is supported by a competitive equilibrium and it is also the maximum CE payoff for buyers related with the minimum CE prices that are $p_1=p_2=0$.

We can provide a sufficient condition in terms of the pairwise valuation matrix that guarantees that all core allocations are supported by competitive prices.

Proposition 14. *Let (B,S,A,r) be a capacity-balanced many-to-one assignment market where buyers have unitary capacity, and μ an optimal matching. Then $C(B)=CE(B)$ if for all $j,j'\in S$ and $i,k,i'\in B$ such that $\mu(i)=\mu(k)=j$ and $\mu(i')=j'\neq j$ it holds*

$$a_{kj'}+a_{i'j}\geq a_{kj}+a_{i'j'}, \tag{9}$$
$$a_{ij'}+a_{i'j}\geq a_{ij}+a_{i'j'}, \tag{10}$$

Proof. Take $x\in C(B)$. From the core constraints, together with (9) and (10), we get

$$x_k-x_i=(x_k-x_{i'})+(x_{i'}-x_i)\geq a_{kj'}-a_{i'j'}+a_{i'j}-a_{ij}\geq a_{kj}-a_{ij},\ \text{and}$$
$$x_i-x_k=(x_i-x_{i'})+(x_{i'}-x_k)\geq a_{ij'}-a_{i'j'}+a_{i'j}-a_{kj}\geq a_{ij}-a_{kj},$$

which proves that $x_k-x_i=a_{kj}-a_{ij}$. □

Our previous characterization (Theorem 6) of the extreme competitive salaries of the multiple-partners job market can be straightforwardly extended to the extreme competitive buyers' payoffs of Kaneko's buyer-seller market, simply defining the tight digraph of a CE payoff vector using all the inequalities in Proposition 12. Take, for instance, the extended tight digraph of the minimum CE payoff vector for buyers in Example 13 (see Figure 4) and notice that it has a unique source.

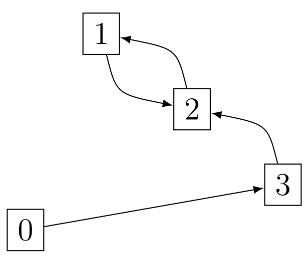

Figure 4: Extended tight digraph of the minimum CE payoff vector (4,2,0).

Similarly, the definition of the max-min vectors in Definition 8 can be modified by the omission of the condition $j^\mu\neq j^\mu_r$. Then, a result analogous to Theorem 11 guarantees that each extreme point of $CE(B)$ coincides with one of these max-min vectors.

7 Concluding remarks

The described procedure of the max-min salary vectors provides all extreme core allocations. Since all orders on the set of workers must be considered, this is not a very efficient procedure. Nevertheless, compared to the well-studied maximum and minimum competitive salary vectors, it allows to find other combinations of competitive salaries where the payoff of some workers is maximized while for others it is minimized, everything according to a given order of priority.

An interesting direction for future research is to characterize core stability using matrix properties. Solymosi and Raghavan (2001) introduced a condition known as the *dominant diagonal property* and used graph-theoretical tools to characterize core stability in one-to-one assignment games. Later, Atay (2017) provided an alternative proof based on the properties of the buyer-seller exact representation of an assignment game, as defined in Núñez and Rafels (2003). Extending this line of work, future research could explore many-to-one assignment games with the dominant diagonal property to examine the robustness of the core.

Appendix A

This example shows that, for many-to-one assignment markets, neither all lemaral vectors are extreme core allocations nor all extreme core points can be obtained as lemaral vectors for some given order on the agents. Consider the market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$ are the set of firms and the set of workers respectively, and the capacities of the firms are $r = (2, 1)$. The per-unit pairwise valuations are given in the following matrix:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{cc} w_1 & w_2 \end{array} \\ \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix} \end{array}.$$

The corresponding many-to-one assignment game (N, v_γ) and its dual game are:

v	x_1	x_2	y_1	y_2	v^*
0	1	.	.	.	4
0	.	1	.	.	0
0	.	.	1	.	4
0	.	.	.	1	3
0	1	1	.	.	7
4	1	.	1	.	5
3	1	.	.	1	4
3	.	1	1	.	4
2	.	1	.	1	3
0	.	.	1	1	7
4	1	1	1	.	7
3	1	1	.	1	7
7	1	.	1	1	7
3	.	1	1	1	7
7	1	1	1	1	7

The marginal payoff of firm f_1 is $4 = v^*(f_1) = v(N) - v(N \setminus f_1)$ but it is not achievable in the core since the core-maximum for firm f_1 is $\max_C x_1 = 2 = v^*(\{f_2\}) + v^*(\{f_1, w_1\}) + v^*(\{f_1, w_2\}) - v^*(N)$. This shows that the marginal payoff of a player to the grand coalition may not be the core maximum payoff of the corresponding player for the many-to-one assignment game.

Now, take any order that starts with the firm f_1, $\sigma = (f_1, arbitrary)$. For that given order, the payoff of f_1 is 4 which cannot be attained at a core allocation. Hence, a lemaral obtained by an order $\sigma = (f_1, arbitrary)$ cannot be a core allocation.

Next, take the extreme core allocation $(2, 0; 3, 2)$. Notice that $\min_C y_2 = 2 = v(\{f_2, w_2\}) + v(\{f_1, w_1, w_2\}) - v(N)$ and both f_1 and f_2 obtain their core maximum allocations, and hence $(2, 0; 3, 2)$ is an extreme core allocation. We will try to construct a lemaral vector $(x, y) \in \mathbb{R}^N$ that coincides with the aforementioned extreme core allocation. First notice that f_2 is the only player that is paid her marginal payoff. Hence, we only take into account orders that start with player f_2:

- Player 2 achieves her marginal payoff under an order $\sigma = (f_2, arbitrary)$: $x_2 = 0$,
- $\sigma = (f_2, f_1, \ldots)$: Then,
$$x_1 = \min\{v^*(f_1), v^*(\{f_1, f_2\}) - x_2\} = \min\{4, 7 - 0\} = 4 \neq 2 = x_1,$$
- $\sigma = (f_2, w_1, \ldots)$: Then,
$$y_1 = \min\{4, 4 - 0\} = 4 \neq 3 = y_1,$$
- $\sigma = (f_2, w_2, \ldots)$: Then,
$$y_2 = \min\{3, 3 - 0\} = 3 \neq 2 = y_2.$$

As a consequence, there does not exist an order to construct a lemaral vector that coincides with the extreme core allocation $(2, 0; 3, 2)$. One can find a similar example to show that some extreme core allocations cannot be obtained as a result of a max-payoff vector (Izquierdo et al., 2007).

Appendix B

All max-min salary vectors in the market from Example 10:

ext. order	y_1	y_2	y_3	in core?
$(\underline{1}, \underline{2}, \underline{3})$	0	0	0	−
$(\underline{1}, \underline{2}, \overline{3})$	0	0	−3	−
$(\underline{1}, \overline{2}, \underline{3})$	0	6	3	−
$(\underline{1}, \overline{2}, \overline{3})$	0	6	−3	−
$(\overline{1}, \underline{2}, \underline{3})$	8	0	3	−
$(\overline{1}, \underline{2}, \overline{3})$	8	0	−2	−
$(\overline{1}, \overline{2}, \underline{3})$	8	6	3	+
$(\overline{1}, \overline{2}, \overline{3})$	8	6	4	+

ext. order	y_1	y_2	y_3	in core?
$(\underline{1}, \underline{3}, \underline{2})$	0	2	0	−
$(\underline{1}, \underline{3}, \overline{2})$	0	3	0	−
$(\underline{1}, \overline{3}, \underline{2})$	0	5	−3	−
$(\underline{1}, \overline{3}, \overline{2})$	0	6	−3	−
$(\overline{1}, \underline{3}, \underline{2})$	8	5	3	+
$(\overline{1}, \underline{3}, \overline{2})$	8	6	3	+
$(\overline{1}, \overline{3}, \underline{2})$	8	6	4	+
$(\overline{1}, \overline{3}, \overline{2})$	8	6	4	+

ext. order	y_1	y_2	y_3	in core?
$(\underline{2}, \underline{1}, \underline{3})$	0	0	0	−
$(\underline{2}, \underline{1}, \overline{3})$	0	0	−3	−
$(\underline{2}, \overline{1}, \underline{3})$	8	0	3	−
$(\underline{2}, \overline{1}, \overline{3})$	8	0	−2	−
$(\overline{2}, \underline{1}, \underline{3})$	0	6	3	−
$(\overline{2}, \underline{1}, \overline{3})$	0	6	4	−
$(\overline{2}, \overline{1}, \underline{3})$	8	6	3	+
$(\overline{2}, \overline{1}, \overline{3})$	8	6	4	+

ext. order	y_1	y_2	y_3	in core?
$(\underline{2}, \underline{3}, \underline{1})$	3	0	0	−
$(\underline{2}, \underline{3}, \overline{1})$	5	0	0	−
$(\underline{2}, \overline{3}, \underline{1})$	1	0	−2	−
$(\underline{2}, \overline{3}, \overline{1})$	3	0	−2	−
$(\overline{2}, \underline{3}, \underline{1})$	6	6	3	+
$(\overline{2}, \underline{3}, \overline{1})$	8	6	3	+
$(\overline{2}, \overline{3}, \underline{1})$	7	6	4	+
$(\overline{2}, \overline{3}, \overline{1})$	8	6	4	+

ext. order	y_1	y_2	y_3	in core?
$(\underline{3}, \underline{1}, \underline{2})$	3	2	0	+
$(\underline{3}, \underline{1}, \overline{2})$	3	3	0	+
$(\underline{3}, \overline{1}, \underline{2})$	5	2	0	+
$(\underline{3}, \overline{1}, \overline{2})$	5	3	0	+
$(\overline{3}, \underline{1}, \underline{2})$	7	6	4	+
$(\overline{3}, \underline{1}, \overline{2})$	7	6	4	+
$(\overline{3}, \overline{1}, \underline{2})$	8	6	4	+
$(\overline{3}, \overline{1}, \overline{2})$	8	6	4	+

ext. order	y_1	y_2	y_3	in core?
$(\underline{3}, \underline{2}, \underline{1})$	3	2	0	+
$(\underline{3}, \underline{2}, \overline{1})$	5	2	0	+
$(\underline{3}, \overline{2}, \underline{1})$	3	3	0	+
$(\underline{3}, \overline{2}, \overline{1})$	5	3	0	+
$(\overline{3}, \underline{2}, \underline{1})$	7	6	4	+
$(\overline{3}, \underline{2}, \overline{1})$	8	6	4	+
$(\overline{3}, \overline{2}, \underline{1})$	7	6	4	+
$(\overline{3}, \overline{2}, \overline{1})$	8	6	4	+

Notice that since workers 1 and 2 are optimally matched to the same firm, thus the difference between their core payoffs is not constrained, whenever they occupy consecutive positions in an extended order the associated max-min vector is the same. Based on this observation, the full enumeration process can be somewhat shortened.

References

Atay, A. (2017), "An alternative proof of the characterization of core stability for the assignment game." *Operations Research Letters*, 45, 217–219.

Baïou, M. and M. Balinski (2000), "The stable admissions polytope." *Mathematical Programming*, 87, 427–439.

Baïou, M. and M. Balinski (2002), "The stable allocation (or ordinal transportation) problem." *Mathematics of Operations Research*, 27, 485–503.

Balinski, M. L. and D. Gale (1990), "On the core of the assignment game." In *Functional Analysis, Optimization and Mathematical Economics*, 274–289, Oxford University Press.

Gul, F. and E. Stachetti (1999), "Walrasian equilibrium with gross substitutes." *Journal of Economic Theory*, 87, 95–124.

Hamers, H., F. Klijn, T. Solymosi, S. Tijs, and J. P. Villar (2002), "Assignment games satisfy the CoMa-property." *Games and Economic Behavior*, 38, 231–239.

Izquierdo, J. M., M. Núñez, and C. Rafels (2007), "A simple procedure to obtain the extreme core allocations of an assignment market." *International Journal of Game Theory*, 36, 17–26.

Kaneko, M. (1976), "On the core and competitive equilibria of a market with indivisible goods." *Naval Research Logistics Quarterly*, 23, 321–337.

Núñez, M. and C. Rafels (2003), "Buyer-seller exactness in the assignment game." *International Journal of Game Theory*, 31, 423–436.

Núñez, M. and C. Rafels (2015), "A survey on assignment markets." *Journal of Dynamics and Games*, 2, 227–256.

Núñez, M. and T. Solymosi (2017), "Lexicographic allocations and extreme core payoffs: the case of assignment games." *Annals of Operations Research*, 254, 211–234.

Quint, T. (1991), "Characterization of cores of assignment games." *International Journal of Game Theory*, 19, 413–420.

Sánchez-Soriano, J., M. A. Lopez, and I. Garcia-Jurado (2001), "On the core of transportation games." *Mathematical Social Sciences*, 41, 215–225.

Shapley, L. and M. Shubik (1971), "The assignment game I: The core." *International Journal of Game Theory*, 1, 111–130.

Solymosi, T. and T. E. S. Raghavan (1994), "An algorithm for finding the nucleolus of assignment games." *International Journal of Game Theory*, 23, 119–143.

Solymosi, T. and T.E.S. Raghavan (2001), "Assignment games with stable core." *International Journal of Game Theory*, 30, 177–185.

Sotomayor, M. (1992), "The multiple partners game." In *Equilibrium and Dynamics*, 322–354, Springer.

Sotomayor, M. (2002), "A labor market with heterogeneous firms and workers." *International Journal of Game Theory*, 31, 269–283.

Sotomayor, M. (2007), "Connecting the cooperative and competitive structures of the multiple-partners assignment game." *Journal of Economic Theory*, 134, 155–174.

Trudeau, C. and J. Vidal-Puga (2017), "On the set of extreme core allocations for minimal cost spanning tree problems." *Journal of Economic Theory*, 169, 425–452.