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A note on the non-coincidence of the core and the bargaining set in many-to-one assignment markets ⋆

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This paper analyzes the extent to which well-known results on the relationship between the bargaining set, the core, and the kernel in one-to-one assignment games generalize to many-to-one assignment markets, and by extension, many-to-many markets. Using a minimal counterexample, we show that the bargaining set does not necessarily coincide with the core and that the kernel may not be contained within the core. We would like to highlight that the failure of the coincidence between the core and the bargaining set, as observed in the many-to-one assignment game, is quite notable. This is especially true when compared to various other highly structured games, many of which emerge from combinatorial optimization problems, such as the one underlying many-to-one assignment games.

1. Introduction

We consider a two-sided market, one side formed by a set of firms and the other by a set of workers. Firms want to hire many workers, up to each firm's capacity, but each worker can work for only one firm. Each firm places a non-negative value on each worker, and workers may have a reservation value. Since we assume that firms value groups of workers additively, the main data of the market is the value that each firm-worker pair can attain when matched. This value can be transferred by means of the salary the firm pays to each worker it hires. A matching is an assignment of a group of workers to each firm, and a coalitional game is introduced where the worth of a coalition is the highest value that can be obtained by matching firms and workers in the coalition without violating the capacity of each firm. A natural solution concept in this setting is the core, which is the set of allocations of the total value of the market that cannot be improved upon by any coalition.

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The many-to-one assignment market is an extension of the well-known one-to-one *assignment game* introduced by Shapley and Shubik (1971) to study two-sided markets where there are indivisible goods which are traded between sellers and buyers in exchange for money. In their model, each buyer wants at most one unit of good, and each seller owns exactly one indivisible good.[1] One-to-one assignment markets can be extended to many-to-many assignment markets in different ways.[2]

In this note we focus on the core of the many-to-one assignment market games and consider other related set-solution concepts different from the core, and defined for the general class of coalitional games, to show that more dissimilarities appear with respect to the one-to-one case. One of these solution concepts, that contains the core, is the classical bargaining set of Davis and Maschler (1967), formed by those imputations that have no justified objection. That is, for an imputation to be in the bargaining set, it is needed that whenever this imputation is objected by a coalition proposing a payoff vector that makes all its members better off, then another coalition exists that can make a counterobjection. The bargaining set contains the core since core elements have no objection (no group of agents can improve their payoffs by breaking away and redistributing their joint worth among themselves). The second set-solution notion we will consider is the kernel (Davis and Maschler, 1965).[3] The kernel refines the bargaining set by requiring a balance in "bargaining power" between every pair of agents: at each kernel imputation each pair of agents equalizes their maximum potential complaint from excluding each other.

For the one-to-one assignment game, first Driessen (1998) proved that the kernel is included in the core, and secondly Solymosi (1999) proved that the bargaining set coincides with the core.[4] We show that, differently from the one-to-one assignment market, the kernel may not be a subset of the core for many-to-one assignment markets. Therefore, the core and the classical bargaining set do not coincide (Example 1 and Proposition 2). The loss of this coincidence means that the core is somehow a less robust solution in these markets, since those allocations outside the core that do not have a justified objection could be taken into account when looking for a distribution of the worth of the grand coalition. Maschler (1976) points out the advantage of the bargaining set over the core in some market games. What we highlight in this note is that this is also the case in some many-to-one assignment markets.

For any coalitional game with a non-empty imputation set, the intersection of the kernel and the core is always non-empty, since it contains a well-known single-valued solution that is the nucleolus. We show that the simplified expression for this intersection that was given in Granot and Granot (1992) for the one-to-one assignment game can now be extended to the many-to-one case.

The paper is organized as follows. In Section 2, some preliminaries on transferable utility games are provided. Section 3 introduces many-to-one assignment markets and games. In Section 4, we conclude with our results on the core, the kernel, and the bargaining set for these many-to-one markets.

2. Notations and definitions

A *transferable utility (TU) cooperative game* (N, v) is a pair where N is a non-empty, finite set of *players (or agents)* and $v : 2^N \to \mathbb{R}$ is a *coalitional function* satisfying $v(\emptyset) = 0$. The number $v(S)$ is regarded as the worth of the coalition $S \subseteq N$. We identify the game with its coalitional function since the player set N is fixed throughout the paper. The game (N, v) is called *superadditive* if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$ for every two coalitions $S, T \subseteq N$. Coalition $R \subseteq N$ is called *inessential* in game v if it has a nontrivial partition $R = S \cup T$ with $S, T \neq \emptyset$ and $S \cap T = \emptyset$ such that $v(R) \leq v(S) + v(T)$. Notice that in a superadditive game the weak majorization can only happen as equality. Those non-empty coalitions which are not inessential are called *essential*. Note that the single-player coalitions are essential in any game, and any inessential coalitional value can be weakly majorized by the value of a partition composed only of essential coalitions.

Given a game (N, v), a *payoff allocation* $x \in \mathbb{R}^N$ represents the payoffs to the players. The total payoff to coalition $S \subseteq N$ is denoted by $x(S) = \sum_{i \in S} x_i$, in particular $x(\emptyset) = 0$, for throughout the paper we keep the convention that summing over the empty-set gives zero. In a game v, we say the payoff allocation x is *efficient*, if $x(N) = v(N)$. The set of *imputations*, denoted by $I(v)$, consists of all efficient payoff vectors that are *individually rational*, that is, $x_i \geq v(\{i\})$ for all $i \in N$. The core $C(v)$ is the set of imputations that are *coalitionally rational*, that is, $x(S) \geq v(S)$ for all $S \subseteq N$. Observe that all the coalitional rationality conditions for inessential coalitions are implied by the inequalities related to essential coalitions, hence can be ignored: the core and the essential-core are always the same.

When a payoff allocation x is not coalitionally rational for a game (N, v), that is $\sum_{i \in S} x_i < v(S)$ for some $S \subseteq N$, we say that coalition S blocks x, since it can propose a payoff vector $y \in \mathbb{R}^N$ such that $y(S) = v(S)$ and $y_k > x_k$ for all $k \in S$. We can also say that (S, y) is an objection of any agent $i \in S$ against any $j \notin S$ at imputation x. The core excludes all those allocations that can be objected. Instead, the (*classical*) *bargaining set*, introduced by Davis and Maschler (1967) for games with a non-empty set of imputations, only excludes allocations with a justified objection, that is, an objection that cannot be countered. A counter-objection to an objection

[1] (Núñez and Rafels, 2015) is a survey on assignment markets and games.

[2] See Sotomayor (1992, 2002, 2007), Sánchez-Soriano et al. (2001), Crawford and Knoer (1981), Kelso and Crawford (1982). In all cases the core is proved to be non-empty but most of the other existing results for the one-to-one assignment markets cannot be extended to the many-to-many case.

[3] These concepts have been applied in voting (Schofield, 1980), matching problems (Klijn and Massó, 2003; Atay et al., 2021), and network flow problems (Granot and Granot, 1992) among others.

[4] Intermediate results were obtained by Granot (1994, 2010) who introduced the reactive bargaining set, a subset of the classical bargaining set, that always contains both the core and the kernel, and proved the coincidence of the core with the reactive bargaining set in one-to-one assignment games.

(S, y) of i against j at $x \in I(v)$, is a pair (T, z) where $j \in T \subseteq N$, $i \notin T$, $y(T) = v(T)$ and $z_k \geq y_k$ for all $k \in T \cap S$ while $z_k \geq x_k$ for all $k \in T \setminus S$.

The core of a coalitional game is always included in its bargaining set, since at core allocations no objection can arise. Hence, the bargaining set often contains infinitely many payoff vectors. We consider another set-wise solution concept known as the kernel. The *kernel*, introduced by Davis and Maschler (1965), is a non-empty subset of the *classical bargaining set* of Davis and Maschler (1967) for all games with a non-empty set of imputations. Whenever the core is non-empty, the intersection between the kernel and the core is also non-empty since it contains the nucleolus.

For superadditive games, the kernel is defined as

$$\mathcal{K}(v) = \{z \in I(v) \mid s_{ij}(z) = s_{ji}(z), \text{ for all } i, j \in N\},$$

where $s_{ij}(z) = \max_{i \in S, j \notin S}\{v(S) - z(S)\}$ is the maximum excess at imputation z of coalitions containing i and not containing j. An imputation in the kernel is pairwise balanced in the sense that the maximum excess that agent i can attain with no cooperation of j equals the maximum excess that j can attain without the cooperation of i.

3. The many-to-one assignment market and game

We consider a market where there are two types of agents: a finite set of firms $F = \{f_1, f_2, \ldots, f_m\}$ and a finite set of workers $W = \{w_1, w_2, \ldots, w_n\}$ where the number of firms m can be different from the number of workers n. Let $N = F \cup W$ be the set of all agents. We sometimes denote a generic firm and a generic worker by i and j, respectively. The net income generated when firm $i \in F$ hires worker $j \in W$ is denoted by a_{ij} and it is shared by means of the salary y_j that the firm pays to the worker. The valuation matrix denoted by $A = (a_{ij})_{(i,j)\in F\times W}$ represents the pairwise income for each possible firm-worker pair. Each firm $i \in F$ would like to hire up to $r_i \geq 0$ workers and each worker $j \in W$ can work for at most one firm. Then, a many-to-one assignment market is the quadruple $\gamma = (F, W, A, r)$.

A *matching* μ for the market $\gamma = (F, W, A, r)$ is a set of $F \times W$ pairs such that each firm $i \in F$ appears in at most r_i pairs and each worker $j \in W$ in at most one pair. We denote by $\mathcal{M}(F, W, r)$ the set of matchings for market γ. A matching $\mu \in \mathcal{M}(F, W, r)$ is *optimal* for γ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ holds for any other matching $\mu' \in \mathcal{M}(F, W, r)$. We denote by $\mathcal{M}_A(F, W, r)$ the set of optimal matchings for the market γ. Given a matching $\mu \in \mathcal{M}(F, W, r)$, the set of workers matched to firm $i \in F$ under μ is $\mu(i) = \{j \in W \mid (i, j) \in \mu\}$. It may be convenient to denote the set of workers unmatched under μ by $\mu(f_0)$, that is $\mu(f_0) = W \setminus \bigcup_{i\in F} \mu(i)$. Observe that $i \neq k \in F$ implies $\mu(i) \cap \mu(k) = \emptyset$, hence $\mu(f_0) \cup \bigcup_{i\in F} \mu(i) = W$ is a partition of the set of workers.

Given a many-to-one assignment market $\gamma = (F, W, A, r)$, we define the income maximization linear programming problem by

$$\begin{aligned} \mathcal{V}(F, W) = \max \ & \sum_{i\in F}\sum_{j\in W} a_{ij}x_{ij} \\ \text{s. t.} \ & \sum_{j\in W} x_{ij} \leq r_i, \quad i \in F \\ & \sum_{i\in F} x_{ij} \leq 1, \quad j \in W \\ & x_{ij} \geq 0, \quad (i, j) \in F \times W. \end{aligned} \tag{1}$$

It is well known that any variable in any basic feasible solution of this LP problem with integer right hand sides is integral, hence, by the worker capacity inequalities, 0 or 1. Consequently, the relation $(i, j) \in \mu \leftrightarrow x_{ij} = 1$ defines a bijection between the set of basic feasible solutions to this LP problem and the set of matchings $\mu \in \mathcal{M}(F, W, r)$. Henceforth, the optimum value of (1) gives the maximum of the sum of values of the matched pairs while respecting the capacities of firms.

Given market $\gamma = (F, W, A, r)$, we also apply the above notation and terminology for any submarket $\gamma_{(S,T)} = (S, T, A_{(S,T)}, r_S)$ with $S \subseteq F$, $T \subseteq W$, and accordingly restricted valuation matrix $A_{(S,T)}$ and capacity vector r_S.

Now, let us associate a coalitional game with transferable utility (TU-game) with this type of two-sided matching markets. Given a many-to-one assignment market $\gamma = (F, W, A, r)$, its associated *many-to-one assignment game* is the pair (N, v_γ) where $N = F \cup W$ is the set of players and the coalitional function is given by $v_\gamma(S \cup T) = \max_{\mu\in\mathcal{M}(S,T,r_S)} \sum_{(i,j)\in\mu} a_{ij}$ for all $S \subseteq F$ and $T \subseteq W$.[5] For brevity, we denote coalition $S \cup T$ with $S \subseteq F$ and $T \subseteq W$ by (S, T), in particular, one-sided coalitions by $(\emptyset, T)$ and $(S, \emptyset)$. As the union of matchings for disjoint coalitions is a matching for the union of the coalitions, i.e. $\mu \in \mathcal{M}(S, T, r_S)$ and $\mu' \in \mathcal{M}(S', T', r_{S'})$ with $S \cap S' = \emptyset$ and $T \cap T' = \emptyset$ implies $\mu \cup \mu' \in \mathcal{M}(S \cup S', T \cup T', r_{S\cup S'})$, it easily follows that many-to-one assignment games are superadditive. On the other hand, if $\nu \in \mathcal{M}(S, T, r_S)$ is an optimal matching for coalition (S, T), that is $v_\gamma(S, T) = \sum_{(i,j)\in\nu} a_{ij} = \sum_{i\in S}\sum_{j\in\nu(i)} a_{ij}$, then it follows from $v_\gamma(i, \nu(i)) = \sum_{j\in\nu(i)} a_{ij}$ for all $i \in S$ that $v_\gamma(S, T) = \sum_{i\in S} v_\gamma(i, \nu(i))$. Since $(S, T) = (\emptyset, \nu(f_0)) \cup \bigcup_{i\in S}(i, \nu(i))$ where $\nu(f_0)$ denotes the unmatched workers in T under ν, and $v_\gamma(\nu(f_0)) = 0 = \sum_{j\in\nu(f_0)} v_\gamma(j)$, we get the following observations.

Proposition 1. *In many-to-one assignment games, the following types of coalitions are inessential:*

- *any coalition containing at least two firms,*

[5] When no confusion arises, for a given market γ, we denote its corresponding coalitional function by v instead of v_γ.

- *any single-firm coalition containing more workers than the capacity of the firm,*
- *any one-sided coalition containing at least two players.*

Consequently, in an $(m+n)$-player many-to-one assignment game, among the $2^{m+n}-1$ non-empty coalitions, at most $\sum_{i=1}^{m}\sum_{t=1}^{r_i}\binom{n}{t} \leq 2^n - 2$ coalitions can be essential. However, this exponential upper bound is sharp (if all a_{ij} pairwise income values are positive, $m=2$, and $n = r_1 + r_2$).

As in any coalitional game, the main concern is how to share the worth of the grand coalition (the total income) among all agents. To do so, we focus on the solution concept known as the *core*. Different than one-to-one assignment games, where the only essential coalitions are the individual ones and the mixed-pairs, here (see Proposition 1), instability may arise from a group of workers and a firm that can be better off by recontracting among themselves instead of their prescribed agreements.

4. Core, kernel and bargaining set

In order to investigate the relationship between the core of many-to-one assignment markets and other set-valued solution concepts, we need to analyze closely the structure of the core of these games. Given a many-to-one assignment market $\gamma = (F, W, A, r)$ and $\mu \in \mathcal{M}_A(F, W, r)$ an optimal matching, $(x, y) \in \mathbb{R}^F_+ \times \mathbb{R}^W_+$ is in the core $C(v_\gamma)$ of the associated game if and only if for every firm $i \in F$,

$$x_i + \sum_{j \in T} y_j \geq \sum_{j \in T} a_{ij} = v_\gamma(i, T) \text{ for all } T \subseteq W \text{ with } |T| \leq r_i (\text{with equality for } T = \mu(i)) \tag{2}$$

and the payoff to unassigned firms or workers is zero.

The above description of the core of a many-to-one assignment game is based on Proposition 1 and the general equivalence of the core and the essential-core.

Other relevant results for the one-to-one assignment game are the coincidence between the core and the bargaining set (Solymosi, 1999) and the inclusion of the kernel in the core (Driessen, 1998), (Granot, 1994, 2010). The coincidence between the core and the bargaining set, when it holds, is a robustness property of the core since it guarantees that any allocation outside the core has a justified objection (an objection that has no counter-objection) and hence can be dismissed when looking for a cooperative agreement on the distribution of the worth of the grand coalition.

In this section, we focus on the relationship between the core and the bargaining set for many-to-one assignment markets. Next example shows that the inclusion of the kernel in the core, and also the coincidence of the core with the bargaining set, do not carry over to the many-to-one assignment game. We remark that this five-player counter-example is of the smallest size possible, since for any balanced game with at most four players the bargaining set and the core coincide (Solymosi, 2002), hence the kernel is included in the core (Peleg, 1966).

Example 1. Consider a many-to-one assignment market $\gamma = (F, W, A, r)$ where $F = \{f_1, f_2\}$ is the set of firms, $W = \{w_1, w_2, w_3\}$ is the set of workers, and the capacities of the firms are $r = (2, 2)$. The per-unit pairwise valuation matrix is the following:

$$A = \begin{array}{c} \\ f_1 \\ f_2 \end{array} \begin{array}{c} \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \end{array}.$$

The core contains a unique point, $C(v_\gamma) = \{(0, 0; 1, 1, 1)\}$. One can easily check that the imputation $(x, y) = (1, 1; 1/3, 1/3, 1/3)$ is not a core allocation, but it lies in the kernel since for any pair of agents the maximum surplus is the same. Let us show that this is indeed the case. The characteristic function of the corresponding many-to-one assignment game $(F \cup W, v_\gamma)$ and the excesses at imputation $(x, y) = (1, 1; 1/3, 1/3, 1/3)$, that is, $e(S, (x, y))$, are represented in the next table:

$v_\gamma(S)$	x_1	x_2	y_1	y_2	y_3	$e(S,(x,y))$
0	1	.	.	.	.	−1
0	.	1	.	.	.	−1
0	.	.	1	.	.	−1/3
0	.	.	.	1	.	−1/3
0	.	.	.	.	1	−1/3
0	1	1	.	.	.	−2
1	1	.	1	.	.	−1/3
1	1	.	.	1	.	−1/3
1	1	.	.	.	1	−1/3
1	.	1	1	.	.	−1/3
1	.	1	.	1	.	−1/3
1	.	1	.	.	1	−1/3
0	.	.	1	1	.	−2/3
0	.	.	1	.	1	−2/3
0	.	.	.	1	1	−2/3

$v_\gamma(S)$	x_1	x_2	y_1	y_2	y_3	$e(S,(x,y))$
1	1	1	1	.	.	−4/3
1	1	1	.	1	.	−4/3
1	1	1	.	.	1	−4/3
2	1	.	1	1	.	1/3
2	1	.	1	.	1	1/3
2	1	.	.	1	1	1/3
2	.	1	1	1	.	1/3
2	.	1	1	.	1	1/3
2	.	1	.	1	1	1/3
0	.	.	1	1	1	−1
2	1	1	1	1	.	−2/3
2	1	1	1	.	1	−2/3
2	1	1	.	1	1	−2/3
2	1	.	1	1	1	0
2	.	1	1	1	1	0
3	1	1	1	1	1	0

Next, we calculate the maximum excess at the point $(x, y) = (1, 1; 1/3, 1/3, 1/3)$ for all possible pairs of agents. Since firms (workers) have the same payoff, $x_1 = x_2 = 1$ ($y_1 = y_2 = y_3 = 1/3$), it is sufficient to check the maximum excess for the following pairs:

- If $i = f_1$ and $j = f_2$, then $\max_{\substack{f_1 \in S \\ f_2 \notin S}} e(S, (x, y)) = 1/3 = \max_{\substack{f_2 \in S \\ f_1 \notin S}} e(S, (x, y))$.
- If $i = w_1$ and $j = w_2$, then $\max_{\substack{w_1 \in S \\ w_2 \notin S}} e(S, (x, y)) = 1/3 = \max_{\substack{w_2 \in S \\ w_1 \notin S}} e(S, (x, y))$.
- If $i = f_1$ and $j = w_1$, then $\max_{\substack{f_1 \in S \\ w_1 \notin S}} e(S, (x, y)) = 1/3 = \max_{\substack{w_1 \in S \\ f_1 \notin S}} e(S, (x, y))$.

Then, for any pair of agents, the maximum excess over coalitions containing one of them but not the other is equal at the point $(x, y) = (1, 1; 1/3, 1/3, 1/3)$. Hence, the imputation $(x, y) = (1, 1; 1/3, 1/3, 1/3)$ lies in the kernel. Since it is not a core allocation, it implies that the kernel is not a subset of the core.

In fact, making use of the facts that (i) the two firms, as well as any two workers are symmetric players[6] in this game, and (ii) symmetric players get the same payoff at any point of the kernel, it can be easily proved that the kernel of this game is the set $\{(\alpha, \alpha; 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3}) \mid 0 \leq \alpha \leq 3/2\}$. It is the line segment with the core element $(0, 0; 1, 1, 1)$ as one extreme point, and the imputation $(3/2, 3/2; 0, 0, 0)$ where the firms take all the value as the other extreme point. Notice that the kernel allows for more fair distributions of the value of the market between firms and workers than the one proposed by the core.[7]

Since the kernel is a subset of the classical bargaining set, all these points $(\alpha, \alpha; 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3}, 1 - \frac{2\alpha}{3})$ that are in the kernel are also in the bargaining set. This implies that the core and the classical bargaining set do not coincide.

Proposition 2. *In the many-to-one assignment game,*

(i) The kernel need not be a subset of the core.
(ii) The classical bargaining set need not coincide with the core.

The above proposition implies that the coincidence between the classical bargaining set and the core cannot be carried over from the one-to-one case to the many-to-one case and to the many-to-many case. Our result showing that the coincidence between the core and the classical bargaining set is not satisfied is a remarkable exception among several classes of related combinatorial optimization games. Solymosi et al. (2003) showed the coincidence result for permutation games. Solymosi (2008) proved (among other variants) that the classical bargaining set coincides with the core for several classes, including one-to-one assignment games, tree-restricted superadditive games, and simple network games. The same coincidence was obtained for monotonic veto games by Solymosi (1999), and extended to any non-negative veto game by Bahel (2016). Recently, Bahel (2021) obtained the coincidence result for the so-called quasi-hyperadditive games, which contain one-to-one assignment games. Atay and Solymosi (2018) extended the coincidence result from one-to-one assignment games to a class of multi-sided assignment games known as the supplier-firm-buyer games.

Although we have learned that the kernel of the many-to-one assignment game may have imputations outside the core, we know that, whenever the core is non-empty, like in the case of the many-to-one assignment games, the kernel always contains some core elements. The reason is that, for games with a non-empty core, it is well-known that the nucleolus is always in the intersection of the kernel and the core.

As in the one-to-one assignment game (Granot and Granot, 1992), some simplifications can be done to obtain those core allocations that also belong to the kernel. First, only essential coalitions are to be taken into account, and secondly, not all pairs of agents need to be considered.

We have defined a matching as a set of firm-worker pairs that do not violate the capacities of firms and workers, but we can also understand it as a partition of $F \cup W$ in essential coalitions. If $(i, j_1), (i, j_2), \dots, (i, j_k)$ are pairs in a matching μ, then the coalition $T = \{i, j_1, j_2, \dots, j_k\}$ is one element of the partition of $F \cup W$ induced by μ. This fact will simplify notations in the next result.

Proposition 3. *Let $\gamma = (F, W, A, r)$ be a many-to-one assignment game. Then,*

$$\mathcal{K}(v_\gamma) \cap C(v_\gamma) = \{z \in C(v_\gamma) \mid s_{ij}(z) = s_{ji}(z) \text{ for all } \{i, j\} \subseteq T \in \Phi(A)\},$$

where $\Phi(A)$ is the set of essential coalitions that belong to all optimal matchings.

Proof. Let $z \in C(v_\gamma)$, S an arbitrary coalition of $F \cup W$ and $\mu_S = \{T_1, T_2, \dots, T_r\}$ an optimal matching for coalition S. Then,

$$e(S, z) = v_\gamma(S) - z(S) = \sum_{T \in \mu_S} (v_\gamma(T) - z(T)) \leq v_\gamma(T_k) - z(T_k), \text{ for all } T_k \in \mu_S,$$

where the inequality follows from the fact that excesses at a core allocation are always non-positive. Then, the maximum excess at z over coalitions containing agent i and not containing agent j is always attained at an essential coalition. This implies that only essential coalitions are to be considered to find those core elements that belong to the kernel.

[6] Two players i and j are symmetric in a coalitional game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

[7] For example, at the kernel-allocation $(3/4, 3/4; 1/2, 1/2, 1/2)$ the shares of the two sides are equal. In this example, the Shapley value (Shapley, 1953) is $(78/120, 78/120; 68/120, 68/120, 68/120)$ and it is also in the kernel of the game.

Moreover, if we take two firms $i_1, i_2 \in F$, then

$$s_{i_1 i_2}(z) = e(S,z) = 0 = e(T,z) = s_{i_2 i_1}(z),$$

where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$, for any optimal matching μ.

Similarly, if we take two workers that are not assigned to the same firm in an optimal matching μ, that is, $(i_1, j_1) \in \mu$ and $(i_2, j_2) \in \mu$, then

$$s_{j_1 j_2}(z) = e(S,z) = 0 = e(T,z) = s_{j_2 j_1}(z),$$

where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$.

Finally, if we take a firm i_1 and a worker j_2 that are not matched in some optimal matching μ, that is, there is $\mu \in \mathcal{M}_A(F,W,r)$ and $i_2 \in F \setminus \{i_1\}$ such that $(i_2, j_2) \in \mu$, then also $s_{i_1 j_2}(z) = e(S,z) = 0 = e(T,z) = s_{j_2 i_1}(z)$, where $S = \{i_1\} \cup \mu(i_1)$ and $T = \{i_2\} \cup \mu(i_2)$.

To sum up, only firm-worker pairs that are matched in all optimal matchings and pairs of workers that are matched to the same firm in all optimal matchings are to be considered. □

As a consequence, if a market (F,W,A,r) is such that there is no essential coalition that belongs to all optimal matchings, then all core elements are in the kernel. This is precisely the case of Example 1.

5. A concluding remark

We would like to highlight that the failure of the coincidence between the core and the bargaining set implied by the failure of the inclusion of the kernel in the core, as observed in the many-to-one assignment game, is quite notable. This is especially true when compared to various other highly structured games, many of which emerge from combinatorial optimization problems, such as the one underlying many-to-one assignment games.

Data availability

No data was used for the research described in the article.

Declaration of interests

None.

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