School Choice with Farsighted Students

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Abstract

We consider priority-based school choice problems with farsighted students. We show that a singleton set consisting of the matching obtained from the Top Trading Cycles (TTC) mechanism is a farsighted stable set. However, the matching obtained from the Deferred Acceptance (DA) mechanism may not belong to any farsighted stable set. Hence, the TTC mechanism provides an assignment that is not only Pareto efficient but also farsightedly stable. Moreover, looking forward three steps ahead is already sufficient for stabilizing the matching obtained from the TTC. In addition, we show that variations of TTC that improve in terms of no justified envy are farsightedly stable, but may require more farsightedness on behalf of students.

Keywords: school choice; top trading cycle; stable sets; farsighted students.
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1 Introduction

Abdulkadiroğlu and Sönmez (2003) formulate the school choice problem of assigning students to schools as a mechanism design problem.[1] Each student has strict preferences over all schools and each school has a strict priority ordering imposed by state or local laws of all students. The outcome of a school choice problem is a matching that assigns schools to students such that each student is assigned at most one school and no school is assigned to more students than its capacity. Two prominent mechanisms used for priority-based matching are the Gale and Shapley's (1962) Deferred Acceptance (DA) mechanism and the Shapley and Scarf's (1974) Top Trading Cycles (TTC) mechanism. Both mechanisms are strategy-proof: truthful preference revelation is a weakly dominant strategy for students.[2] On the one hand, the TTC mechanism is Pareto efficient while the DA mechanism may select an inefficient matching. On the other hand, the DA mechanism is stable while the TTC mechanism may select an unstable matching.

A stable matching in the context of school choice eliminates justified envy in the sense that there is no unmatched student-school pair (i, s) where student i prefers school s to her assignment and she has higher priority than some other student who is assigned a seat at school s. Since only the preferences of students matters in the context of school choice, the stable matching that results from the DA Mechanism Pareto dominates any other matching that eliminates justified envy and is strategy-proof. However, this matching may still be Pareto-dominated.[3] A Pareto efficient and strategy-proof matching is obtained by the TTC mechanism. There is no mechanism that is both Pareto efficient and stable.[4]

Experimental and empirical studies suggest that individuals often differ in their degree of farsightedness, i.e., their ability to forecast how others will react to the

[1]Abdulkadiroğlu and Anderson (2023) provide an extensive survey of school choice. See also Roth and Sotomayor (1990) or Haeringer (2017) for an introduction to matching problems.

[2]Reny (2022) introduces the Priority-Efficient (PE) mechanism that always selects a Pareto efficient matching that dominates the DA stable matching, but PE is not strategy-proof. Another attempt to improve the efficiency of the DA mechanism can be found in Kesten (2010).

[3]Doğan and Ehlers (2021) characterize the priority profiles for which there exists a Pareto improvement over the DA matching that is minimally unstable among Pareto efficient matchings.

[4]See e.g. Roth (1982). Che and Tercieux (2019) show that both Pareto efficiency and stability can be achieved asymptotically using DA and TTC mechanisms when agents have uncorrelated preferences.

decisions they take. Recent experiments on network formation provide evidence in favour of a mixed population consisting of both myopic and (limited) farsighted individuals (see Kirchsteiger, Mantovani, Mauleon and Vannetelbosch, 2016; Teteryatnikova and Tremewan, 2020). The degree of farsightedness of an individual is likely to be correlated with her level of education or cognitive ability (see Mauersberger and Nagel, 2018). Basteck and Mantovani (2018) test in the lab subjects' cognitive ability and compare their allocation to schools under the Immediate Acceptance (IA) and the Deferred Acceptance mechanisms. They show that, under the manipulable IA mechanism, subjects of high cognitive ability earn higher payoffs than low ability subjects and that substantial ability segregation may result, with the top school enrolling up to 45 percent more high ability students than the worst school. The aim of this paper is to provide a theoretical study of how the presence of farsighted students affects the stability of different mechanisms used for priority-based matching problems.

Up to now, it has been assumed that all students are myopic when they decide to join or leave some school. Myopic students do not anticipate that other students may react to their decisions. Coalitions of farsighted students can anticipate the actions of other students and consider the end matching that their deviations may lead to. For instance, looking forward joining her favourite school s' whose capacity is full, a farsighted student i may join first the school s where she has priority and thereby pushes student j out of school s. Later on, she can exchange her priority at school s with another student k who has priority at school s', prefers s to s' and is worse ranked than j at s. In the end matching both students i and k end up with their favourite school. In the context of school choice, our paper is the first to study the stable matchings when students are farsighted.

Farsighted behavior on one side of the market could be observed in several priority-based matching problems. Public school teachers in France can apply every year to be transferred to another school. The transfer is done through a centralized mechanism where teachers report a list of preferences over schools and priority rules determine who gets what. Priorities are based on a score with criteria set by law that vary over time depending on seniority but also, for instance, if a teacher has taught 5 years in a disadvantaged school (Combe, Tercieux and Terrier, 2022). In this case, when a teacher must decide to apply to transfer from one school to another during her career, she takes into account how such decision and her former experience will impact her future score, and thus her chances for later transfers. A patient in need of a kidney faces several options for treatment. One can wait to receive an offer of a deceased donor transplant, or one can rely on a compatible or incompatible living donor and join a Kidney Exchange Program to exchange her donor for a more compatible donor from another incompatible pair (Ashlagi and Roth, 2021). The procedure for allocating deceased donor kidneys prioritizes certain types of patients, such as young or hypersensitized ones. The availability of a compatible or incompatible living donor impacts the decision of a patient on whether to accept a deceased donor kidney offer and her score in the priority list. In student placement to high schools in New York City, many students participate in an appeals process to be assigned to a school they like more than the prescribed assignment (Abdulkadiroğlu, Pathak and Roth, 2005, 2009). For the academic year 2003-2004, half of the appeals were granted by the Department of Education and about 300 appeals out of about 5,000 were from students who received their stated first choices (Kojima, 2011). This suggests that the possibility of rematching due to this appeals process affects the strategic behavior of some students.

Does the TTC mechanism lead to a stable matching when students become farsighted? To address this question, we adopt the notion of farsighted stable set for school choice problems to study the matchings that are stable when students farsightedly apply to schools while schools myopically and mechanically enroll students.[5] A farsighted improving path for school choice problems consists of a sequence of matchings that can emerge when farsighted students form or destroy matches based on the improvement the end matching offers them relative to the current one while myopic schools always accept any student on their priority lists unless they have full capacity. In the case of full capacity, a school accepts to replace the current match by another match if each student who leaves the school is replaced by a newly enrolled student who has a higher priority. A set of matchings is a farsighted stable set if it satisfies (Internal Stability) for any two matchings belonging to the set, there is no farsighted improving path connecting from one matching to the other one, and (External Stability) there always exists a farsighted improving path from every matching outside the set to some matching within the set.

[5]See Chwe (1994), Mauleon, Vannetelbosch and Vergote (2011), Ray and Vohra (2015, 2019), Herings, Mauleon and Vannetelbosch (2019, 2020), Luo, Mauleon and Vannetelbosch (2021) for definitions of the farsighted stable set.

We show that, once students are farsighted, the matching obtained from the TTC algorithm becomes stable, and moreover, a singleton set consisting of the TTC matching is a farsighted stable set. In fact, we construct a farsighted improving path from any matching leading to the TTC matching. Along the farsighted improving path, students belonging to cycles sequentially act in the order of the formation of cycles in the TTC algorithm. Looking forward towards the end matching (i.e. the TTC matching), students belonging to a cycle first get a seat at the school they have priority. Second, they leave that school and thereby guarantee a free seat at that school. Third, they join the school they are matched to in the TTC matching.

Thus, the matching obtained from the TTC algorithm is not only Pareto efficient and strategy-proof, it is also farsightedly stable. On the contrary, the matching obtained from the DA algorithm may not belong to any farsightedly stable set. In addition, starting from any matching, students only need to look forward (at least) three steps ahead to have incentives for engaging a move towards the matches they have in the matching obtained from the TTC algorithm. Hence, little farsightedness is already sufficient for stabilizing the matching obtained from the TTC algorithm.

Hakimov and Kesten (2018) introduce the the Equitable Top Trading Cycles (ETTC) mechanism, a variation of the TTC mechanism for selecting a matching that intends to be more equitable or fair by eliminating avoidable justified envy situations. We show that a singleton set consisting of the ETTC is a farsighted stable set. However, compared to the TTC, the ETTC requires more farsightedness on behalf of students; i.e., students need to look forward more than three steps ahead to have incentives to move towards their ETTC partners. Morrill (2015) proposes both the First Clinch and Trade (FCT) mechanism and the Clinch and Trade (CT) mechanism in order to reduce the distortions the TTC may cause regarding the elimination of justified envy. We show that the matchings obtained from those two variations are farsightedly stable too whenever students belonging to a cycle look forward at least three steps ahead. The TTC algorithm as well as its three variations lead to Pareto efficient matchings. One may be tempted to infer that any Pareto efficient matching can be stabilized once students are farsighted. However, we show that Pareto efficiency is not a sufficient condition for a matching to be farsightedly stable.[6] Notice that the TTC algorithm and these three variations lead also to

[6]The matching obtained from the Immediate Acceptance (IA) algorithm (i.e. the Boston mechanism) may not belong to any farsighted stable set. The IA mechanism satisfies Pareto efficiency

strategy-proof matchings. Unfortunately, Pareto efficiency and strategy-proofness do not guarantee that the outcome of a mechanism belong to a farsighted stable set.

To sum up, farsightedness stabilizes the matching obtained from the TTC algorithm while destabilizes the matching obtained from the DA algorithm, and so may tip the balance in favor of TTC or one of its variations.

In addition, Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2020) provide both theoretical and empirical results supporting the TTC mechanism over alternative mechanisms. The TTC mechanism is justified envy minimal in the class of Pareto efficient and strategy-proof mechanisms in priority-based one-to-one matching problems. Justified envy minimal means that the mechanism satisfies Pareto efficiency with the minimal amount of (myopic) instability. In priority-based many-to-one matching problems, the TTC mechanism admits less justified envy than the Serial Dictatorship mechanism in an average sense. Recently, Doğan and Ehlers (2022) show that, for any stability comparison satisfying three basic properties, the TTC mechanism is minimally unstable among Pareto efficient and strategy-proof mechanisms when schools have unit capacities.

The paper is organized as follows. In Section 2, we introduce priority-based school choice problems. In Section 3, we provide a formal description of the TTC mechanism and its algorithm. In Section 4, we introduce the notions of farsighted improving path and farsighted stable set for school choice problems, and we provide our main result. In Section 5, we look at how much farsightedness is needed for getting our main result. In Section 6, we consider variations of the TTC mechanism. In Section 7, we conclude.

2 School choice problems

A school choice problem is a list $\langle I, S, q, P, F \rangle$ where

(i) $I = \{i_1, ..., i_n\}$ is the set of students,

(ii) $S = \{s_1, ..., s_m\}$ is the set of schools,

(iii) $q = (q_{s_1}, ..., q_{s_m})$ is the quota vector where q_s is the number of available seats at school s,

but is not strategy-proof.

(iv) $P = (P_{i_1}, ..., P_{i_n})$ is the preference profile where P_i is the strict preference of student i over the schools and her outside option,

(v) $F = (F_{s_1}, ..., F_{s_m})$ is the strict priority structure of the schools over the students.

Let i be a generic student and s be a generic school. We write i for singletons $\{i\} \subseteq I$ and s for singletons $\{s\} \subseteq S$. The preference P_i of student i is a linear order over $S \cup i$. Student i prefers school s to school s' if sP_is'. School s is acceptable to student i if sP_ii. We often write $P_i = s, s', s''$ meaning that student i's most preferred school is s, her second best is s', her third best is s'' and any other school is unacceptable for her. Let R_i be the weak preference relation associated with the strict preference relation P_i.[7]

The priority F_s of school s is a linear order over I. That is, F_s assigns ranks to students according to their priority for school s. The rank of student i for school s is denoted $F_s(i)$ and $F_s(i) < F_s(j)$ means that student i has higher priority for school s than student j. For $s \in S, i \in I$, let $F^+(s,i) = \{j \in I \mid F_s(j) < F_s(i)\}$ be the set of students who have higher priority than student i for school s.

A matching μ for a school choice problem is a collection of pairs $\{(i,j)\}_{i \in I, j \in S \cup \{i\}}$ such that for any $i \in I$ and any $s \in S$, (i) $\mu(i) = j \Leftrightarrow (i,j) \in \mu$ where either $j = s \in S$ or $j = i$, (ii) $\mu(s) = \{j \in I \mid (j,s) \in \mu\} \in 2^I$, (iii) $\mu(i) = s \Leftrightarrow i \in \mu(s)$, (iv) $\#\mu(s) \leq q_s$. Condition (i) means that student i is assigned a seat at school s under μ if $\mu(i) = s$ and is unassigned under μ if $\mu(i) = i$. Condition (iv) requires that no school exceeds its quota under μ. That is, for any $s \in S$, we have $\#\mu(s) = \#\{i \in I \mid \mu(i) = s\} \leq q_s$. The set of all matchings is denoted $\mathcal{M}$.[8] For instance, $\mu = \{(i_1, s_2), (i_2, s_1), (i_3, s_1), (i_4, i_4)\}$ is the matching where student i_1 is assigned to school s_2, students i_2 and i_3 are assigned to school s_1 and student i_4 is unassigned.

Given a school choice problem $\langle I, S, q, P, F \rangle$, a matching μ is stable if

(i) for all $i \in I$ we have $\mu(i)R_ii$ (individual rationality),

(ii) for all $i \in I$ and all $s \in S$, if $sP_i\mu(i)$ then $\#\{j \in I \mid \mu(j) = s\} = q_s$ (non-wastefulness),

[7]Haeringer and Klijn (2009) investigate constrained school choice problems where students can only rank a fixed number of schools.

[8]Throughout the paper we use the notation $\subseteq$ for weak inclusion and $\subset$ for strict inclusion. Finally, $\#$ will refer to the notion of cardinality.

(iii) for all $i, j \in I$ with $\mu(j) = s$, if $\mu(j)P_i\mu(i)$ then $j \in F^+(s,i)$ (no justified envy).

Let $\mathcal{S}(I, S, q, P, F)$ be the set of stable matchings. A matching μ' Pareto dominates a matching μ if $\mu'(i)R_i\mu(i)$ for all $i \in I$ and $\mu'(j)P_j\mu(j)$ for some $j \in I$. A matching is Pareto efficient if it is not Pareto dominated by another matching. Let $\mathcal{E}(I, S, q, P, F)$ be the set of Pareto efficient matchings.

A mechanism systematically selects a matching for any given school choice problem (I, S, q, P, F). A mechanism is individually rational (non-wasteful / stable / Pareto efficient) if it always selects an individually rational (non-wasteful / stable / Pareto efficient) matching. A mechanism is strategy-proof if no student can ever benefit by unilaterally misrepresenting her preferences.

3 The Top Trading Cycles algorithm

Abdulkadiroğlu and Sönmez (2003) introduce the Top Trading Cycles (TTC) mechanism for selecting a matching for each school problem. The TTC mechanism finds a matching by means of the following TTC algorithm.

Step 1. Set $q_s^1 = q_s$ for all $s \in S$ where q_s^1 is equal to the initial capacity of school s at Step 1. Each student $i \in I$ points to the school that is ranked first in P_i. If there is no such school, then student i points to herself and she forms a self-cycle. Each school $s \in S$ points to the student that has the highest priority in F_s. Since the number of students and schools are finite, there is at least one cycle. A cycle is an ordered list of distinct schools and distinct students $(s^1, i^1, s^2, ..., s^l, i^l)$ where s^1 points to i^1 (denoted $s^1 \mapsto i^1$), i^1 points to s^2 ($i^1 \mapsto s^2$), $\ldots$, s^l points to i^l ($s^l \mapsto i^l$) and i^l points to s^1 ($i^l \mapsto s^1$). Each school (student) can be part of at most one cycle. Every student in a cycle is assigned a seat at the school she points to and she is removed. Similarly, every student in a self-cycle is not assigned to any school and is removed. If a school s is part of a cycle, then its remaining capacity $q_s^2 = q_s^1 - 1$. If a school s is not part of any cycle, then its remaining capacity $q_s^2 = q_s^1$. If $q_s^2 = 0$, then school s is removed. Let $C_1 = \{c_1^1, c_1^2, ..., c_1^{L_1}\}$ be the set of cycles in Step 1 (where $L_1 \geq 1$ is the number of cycles in Step 1). Let I_1 be the set of students who are assigned to some school at Step 1. Let m_1^l be all the matches from cycle c_1^l that are formed in Step 1 of the algorithm:

$$m_1^l = \begin{cases} \{(i,s) \mid i, s \in c_1^l \text{ and } i \mapsto s\} & \text{if} \quad c_1^l \neq (j) \\ \{(j,j)\} & \text{if} \quad c_1^l = (j) \end{cases} \tag{1}$$

where (j,j) simply means that student j who is in a self-cycle ends up being definitely unassigned to any school. Let $M_1 = \bigcup_{l=1}^{L_1} m_1^l$ be all the matches between students and schools formed in Step 1 of the algorithm.

Step $k \geq 2$. Notice that q_s^k keeps track of how many seats are still available at the school at Step k of the algorithm. Each remaining student $i \in I \setminus \bigcup_{l=1}^{k-1} I_l$ points to the school s that is ranked first in P_i such that $q_s^k \geq 1$. If there is no such school, then student i points to herself and she forms a self-cycle. Each school $s \in S$ such that $q_s^k \geq 1$ points to the student $j \in I \setminus \bigcup_{l=1}^{k-1} I_l$ that has the highest priority in F_s. There is at least one cycle. Every student in a cycle is assigned a seat at the school she points to and she is removed. Similarly, every student in a self-cycle is not assigned to any school and is removed. If a school s is part of a cycle, then its remaining capacity $q_s^{k+1} = q_s^k - 1$. If a school s is not part of any cycle, then its remaining capacity $q_s^{k+1} = q_s^k$. If $q_s^{k+1} = 0$, then school s is removed. Let $C_k = \{c_k^1, c_k^2, ..., c_k^{L_k}\}$ be the set of cycles in Step k (where $L_k \geq 1$ is the number of cycles in Step k). Let I_k be the set of students who are assigned to some school at Step k.

Let m_k^l be all the matches from cycle c_k^l that are formed in Step k of the algorithm:

$$m_k^l = \begin{cases} \{(i,s) \mid i, s \in c_k^l \text{ and } i \mapsto s\} & \text{if} \quad c_k^l \neq (j) \\ \{(j,j)\} & \text{if} \quad c_k^l = (j) \end{cases} \tag{2}$$

Let $M_k = \bigcup_{l=1}^{L_k} m_k^l$ be all the matches between students and schools formed in Step k of the algorithm.

End. The algorithm stops when all students have been removed. Let $\bar{k}$ be the step at which the algorithm stops. Let μ^T denote the matching obtained from the Top Trading Cycles mechanism and it is given by $\mu^T = \bigcup_{k=1}^{\bar{k}} M_k$.

Notice that, for any $k' \in \{1, \ldots, \bar{k} - 1\}$, given all the matches already settled, i.e. $\bigcup_{k=1}^{k'} M_k$, students involve in cycle $c_{k'+1}^l$, $l \in \{1, \ldots, L_{k'+1}\}$, of Step $k'+1$ of the TTC algorithm obtains their best possible assignment in $m_{k'+1}^l$.

Abdulkadiroğlu and Sönmez (2003) show that the TTC mechanism is Pareto efficient and strategy-proof. TTC is also individually rational and non-wasteful, but it is not stable.

In addition to TTC, two alternative mechanisms are also central to the theory of school choice and commonly adopted all over the world: the Deferred Acceptance (DA) algorithm and the Immediate Acceptance (IA) algorithm, also known as the Boston mechanism. Let μ^D denote the matching obtained from the DA mechanism and μ^B denote the matching obtained from the IA (or Boston) mechanism.

4 Farsighted Stable Sets for School Choice

We adopt the notion of farsighted stable set for school choice problems to study the matchings that are stable when students farsightedly apply to schools while schools myopically and mechanically enroll students. The notion of a farsighted stable set for school choice problems is adapted from the notion of a myopic-farsighted stable set that has been introduced by Herings, Mauleon and Vannetelbosch (2020) for two-sided matching problems and by Luo, Mauleon and Vannetelbosch (2021) for network formation games.[9]

A farsighted improving path for school choice problems is a sequence of matchings that can emerge when farsighted students form or destroy matches based on the improvement the end matching offers them relative to the current one while myopic schools form or destroy matches based on the improvement the next matching in the sequence offers them relative to the current one.

Let $\mathcal{P}(\mu(s))$ denote the power set of the set $\mu(s)$, i.e. the set of all subsets of $\mu(s)$.

Definition 1. Given a matching μ, a coalition $N \subseteq I \cup S$ is said to be able to enforce a matching μ' over μ if the following conditions hold:

(i) $\mu'(s) \notin \mathcal{P}(\mu(s)) \cup \{s\}$ implies $\mu'(s) \setminus \mu(s) \cup \{s\} \subseteq N$ and

(ii) $\mu'(s) \in \mathcal{P}(\mu(s)) \cup \{s\}$, $\mu'(s) \neq \mu(s)$, implies either s or $\mu(s) \setminus \mu'(s)$ or s together with a non-empty subset of $\mu(s) \setminus \mu'(s)$ should be in N.

[9]When all agents are myopic, the myopic-farsighted stable set boils down to the pairwise CP vNM set as defined in Herings, Mauleon and Vannetelbosch (2017) for two-sided matching problems. Ehlers (2007) introduces another set-valued concept based upon the concept of vNM stable sets.

Condition (i) says that any new match in μ' that contains different partners than in μ should be such that s and the different partners of s belong to N. Condition (ii) states that so as to leave some (or all) positions of one existing match in μ unfilled, either s or the students leaving such positions or s and some non-empty subset of such students should be in N.

Definition 2. Let $\langle I, S, q, P, F \rangle$ be a school choice problem. A farsighted improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, \ldots, L-1\}$ there is a coalition $N_l \subseteq I \cup S$ that can enforce μ_{l+1} from μ_l and

(i) $\mu_L(i) R_i \mu_l(i)$ for all $i \in N_l \cap I$ and $\mu_L(j) P_j \mu_l(j)$ for some $j \in N_l \cap I$,

(ii) For every $s \in N_l \cap S$ such that $\#\mu_l(s) + \#\{i \in I \mid i \notin \mu_l(s), i \in \mu_{l+1}(s)\} > q_s$, there is $\{i_1, \ldots, i_J\} \subseteq \{i \in I \mid i \notin \mu_l(s), i \in \mu_{l+1}(s)\}$ and $\{j_1, \ldots, j_J\} = \{i \in I \mid i \in \mu_l(s), i \notin \mu_{l+1}(s)\}$ such that

$$\begin{aligned} F_s(i_1) &< F_s(j_1) \\ F_s(i_2) &< F_s(j_2) \\ &\vdots \\ F_s(i_J) &< F_s(j_J). \end{aligned}$$

Notice that $\mu_l(s)$ are the students who are assigned to school s in μ_l and $\{i \in I \mid i \notin \mu_l(s), i \in \mu_{l+1}(s)\}$ are the students who join school s in μ_{l+1}. Thus, a farsighted improving path for school choice problems consists of a sequence of matchings where along the sequence (i) students form or destroy matches based on the improvement the end matching offers them relative to the current one while (ii) schools always accept any student on their priority lists unless they have full capacity. In the case of full capacity, a school $s \in N_l \cap S$ accepts to replace the match μ_l by μ_{l+1} if each student $i \in \{j \in I \mid j \in \mu_l(s), j \notin \mu_{l+1}(s)\}$ who leaves or is evicted from school s from μ_l to μ_{l+1} is replaced by a newly enrolled student who has a higher priority.

Let some $\mu \in \mathcal{M}$ be given. If there exists a farsighted improving path from a matching μ to a matching μ', then we write $\mu \to \mu'$. The set of matchings $\mu' \in \mathcal{M}$ such that there is a farsighted improving path from μ to μ' is denoted by $\phi(\mu)$, so $\phi(\mu) = \{\mu' \in \mathcal{M} \mid \mu \to \mu'\}$.

Definition 3. Let $\langle I, S, q, P, F \rangle$ be a school choice problem. A set of matchings $V \subseteq \mathcal{M}$ is a farsighted stable set if it satisfies:

(i) For every $\mu, \mu' \in V$, it holds that $\mu' \notin \phi(\mu)$.

(ii) For every $\mu \in \mathcal{M} \setminus V$, it holds that $\phi(\mu) \cap V \neq \emptyset$.

Condition (i) of Definition 3 corresponds to internal stability (IS). For any two matchings μ and μ' in the farsighted stable set V there is no farsighted improving path connecting μ to μ'. Condition (ii) of Definition 3 expresses external stability (ES). There always exists a farsighted improving path from every matching μ outside the farsighted stable set V to some matching in V.[10]

Theorem 1. *Let $\langle I, S, q, P, F \rangle$ be a school choice problem and μ^T be the matching obtained from the Top Trading Cycles mechanism. The singleton set $\{\mu^T\}$ is a farsighted stable set.*

Proof. Since $\{\mu^T\}$ is a singleton set, internal stability (IS) is satisfied. (ES) Take any matching $\mu \neq \mu^T$, we need to show that $\phi(\mu) \ni \mu^T$. We build in steps a farsighted improving path from μ to μ^T. Remember that $\mu^T = \bigcup_{k=1}^{\bar{k}} M_k$ where $M_k = \bigcup_{l=1}^{L_k} m_k^l$ are all the matches between students and schools formed in Step k of the TTC algorithm, and m_k^l is given by Expressions (1) and (2).

Step 1.1. If $m_1^1 \subseteq \mu$ and $1 \neq L_1$ then go to Step 1.2 with $\mu'''_{1,1} = \mu$. If $m_1^1 \subseteq \mu$ and $1 = L_1$ then go to Step 1.End with $\mu'''_{1,L_1} = \mu$. If $m_1^1 \nsubseteq \mu$ then $\mu'_{1,1} = \mu - \{(i, \mu(i)) \mid (i, \mu^T(i)) \in m_1^1 \text{ and } \mu(i) \neq i\} + \{(i, s) \mid i, s \in c_1^1 \text{ and } s \mapsto i\} - \{(j, s) \in \mu \mid s \in c_1^1,\ \mu(s) \cap c_1^1 = \emptyset,\ \#\mu(s) = q_s \text{ and } F_s(j) > F_s(j') \text{ for all } j' \in \mu(s), j' \neq j\}$.[11] That is, starting from μ, looking forward towards μ^T, the coalition of students belonging to c_1^1 has incentives to deviate to $\mu'_{1,1}$ where each student in c_1^1 is assigned to the school where she has the highest priority. Students belonging to c_1^1 obtain in μ^T their best possible match. Schools have incentives to accept those students because either they do not have full capacity or the new student replaces the student who had the lowest priority among the students enrolled at the school. Next, students belonging to c_1^1 leave their school to reach $\mu''_{1,1} =$

[10] When all agents (schools and students) are farsighted, the notion of the farsighted stable set in Definition 3 coincides with the definition of the vNM farsighted stable set of Mauleon, Vannetelbosch and Vergote (2011) who define and characterize the vNM farsighted stable set for two-sided matching problems. Doğan and Ehlers (2023) show the existence of myopic-farsighted stable sets for matching problems where there are farsighted agents only on one side of the market, while there may be myopic agents on both sides.

[11] We use the notation $+$ for adding pairs and $-$ for deleting pairs.

$\mu'_{1,1} - \{(i,s) \mid i,s \in c_1^1 \text{ and } s \mapsto i\}$. Next, each student belonging to c_1^1 joins her most preferred school to reach $\mu'''_{1,1} = \mu''_{1,1} + \{(i,s) \mid i,s \in c_1^1 \text{ and } i \mapsto s\}$. Schools accept those students since they have (at least) one vacant position. We reach $\mu'''_{1,1}$ with $m_1^1 \subseteq \mu'''_{1,1}$ and so students belonging to c_1^1 are assigned to the same school as in μ^T. If $1 \neq L_1$, then go to Step 1.2. Otherwise, go to Step 1.End with $\mu'''_{1,L_1} = \mu'''_{1,1}$.

Step 1.l. ($l > 1$) If $m_1^l \subseteq \mu'''_{1,l-1}$ and $l \neq L_1$ then go to Step 1.l+1 with $\mu'''_{1,l} = \mu'''_{1,l-1}$. If $m_1^l \subseteq \mu'''_{1,l-1}$ and $l = L_1$ then go to Step 1.End with $\mu'''_{1,L_1} = \mu'''_{1,l-1}$. If $m_1^l \not\subseteq \mu'''_{1,l-1}$ then $\mu'_{1,l} = \mu'''_{1,l-1} - \{(i, \mu'''_{1,l-1}(i)) \mid (i, \mu^T(i)) \in m_1^l \text{ and } \mu'''_{1,l-1}(i) \neq i\} + \{(i,s) \mid i,s \in c_1^l \text{ and } s \mapsto i\} - \{(j,s) \in \mu'''_{1,l-1} \mid s \in c_1^l,\ \mu'''_{1,l-1}(s) \cap c_1^l = \emptyset,\ \#\mu'''_{1,l-1}(s) = q_s \text{ and } F_s(j) > F_s(j') \text{ for all } j' \in \mu'''_{1,l-1}(s), j' \neq j\}$. From $\mu'''_{1,l-1}$, looking forward towards μ^T, the coalition of students belonging to c_1^l has incentives to deviate to $\mu'_{1,l}$ where each student in c_1^l is assigned to the school where she has the highest priority. Indeed, students belonging to c_1^l obtain in μ^T their best possible match. Schools have incentives to accept those students because either they do not have full capacity or the new student replaces the student who had the lowest priority among the students enrolled at the school. Next, students belonging to c_1^l leave their school to reach $\mu''_{1,l} = \mu'_{1,l} - \{(i,s) \mid i,s \in c_1^l \text{ and } s \mapsto i\}$. Next, each student belonging to c_1^l joins her most preferred school to reach $\mu'''_{1,l} = \mu''_{1,l} + \{(i,s) \mid i,s \in c_1^l \text{ and } i \mapsto s\}$. Schools accept those students since they have (at least) one vacant position. We reach $\mu'''_{1,l}$ with $m_1^l \subseteq \mu'''_{1,l}$ and so students belonging to c_1^l are assigned to the same school as in μ^T. If $l \neq L_1$, then go to Step 1.l+1. Otherwise, go to Step 1.End with $\mu'''_{1,L_1} = \mu'''_{1,l}$.

Step 1.End. We have reached μ'''_{1,L_1} with $\bigcup_{l=1}^{L_1} m_1^l = M_1 \subseteq \mu'''_{1,L_1}$. If $\mu'''_{1,L_1} = \mu^T$ then the process ends. Otherwise, go to Step 2.1.

Step k.1. ($k \geq 2$) If $m_k^1 \subseteq \mu'''_{k-1,L_{k-1}}$ and $1 \neq L_k$ then go to Step k.2 with $\mu'''_{k,1} = \mu'''_{k-1,L_{k-1}}$. If $m_k^1 \subseteq \mu'''_{k-1,L_{k-1}}$ and $1 = L_k$ then go to Step k.End with $\mu'''_{k,L_k} = \mu'''_{k-1,L_{k-1}}$. If $m_k^1 \not\subseteq \mu'''_{k-1,L_{k-1}}$ then $\mu'_{k,1} = \mu'''_{k-1,L_{k-1}} - \{(i, \mu'''_{k-1,L_{k-1}}(i)) \mid (i, \mu^T(i)) \in m_k^1 \text{ and } \mu'''_{k-1,L_{k-1}}(i) \neq i\} + \{(i,s) \mid i,s \in c_k^1 \text{ and } s \mapsto i\} - \{(j,s) \in \mu'''_{k-1,L_{k-1}} \mid s \in c_k^1,\ \mu'''_{k-1,L_{k-1}}(s) \cap c_k^1 = \emptyset,\ \#\mu'''_{k-1,L_{k-1}}(s) = q_s \text{ and } F_s(j) > F_s(j') \text{ for all } j' \in \mu'''_{k-1,L_{k-1}}(s), j' \neq j\}$. Starting from $\mu'''_{k-1,L_{k-1}}$, looking forward towards μ^T, the coalition of students belonging to c_k^1 has now incentives to deviate to $\mu'_{k,1}$

where each student in c_k^1 is assigned to the school where she has the highest priority among students belonging to $I \setminus (\bigcup_{k'=1}^{k-1} I_{k'})$. Remember that $\bigcup_{k'=1}^{k-1} I_{k'}$ is the set of students who are involved in $\bigcup_{k'=1}^{k-1} M_{k'}$. Given that the matches $\bigcup_{k'=1}^{k-1} M_{k'} \subseteq \mu'''_{k-1,L_{k-1}}$ are already settled and remain fixed, students belonging to c_k^1 obtain in μ^T their best possible match. Schools have incentives to accept those students because either they do not have full capacity or the new student replaces the student who had the lowest priority among the students enrolled at the school. Next, students belonging to c_k^1 leave their school to reach $\mu''_{k,1} = \mu'_{k,1} - \{(i,s) \mid i,s \in c_k^1 \text{ and } s \mapsto i\}$. Next, each student belonging to c_k^1 joins her most preferred school (constrained to $\bigcup_{k'=1}^{k-1} M_{k'}$ being fixed) to reach $\mu'''_{k,1} = \mu''_{k,1} + \{(i,s) \mid i,s \in c_k^1 \text{ and } i \mapsto s\}$. Schools accept those students since they have (at least) one vacant position. We reach $\mu'''_{k,1}$ with $m_k^1 \subseteq \mu'''_{k,1}$ and so students belonging to c_k^1 are assigned to the same school as in μ^T. If $1 \neq L_k$, then go to Step k.2. Otherwise, go to Step k.End with $\mu'''_{k,L_k} = \mu'''_{k,1}$.

Step $k.l.$ ($l \geq 2$) If $m_k^l \subseteq \mu'''_{k,l-1}$ and $l \neq L_k$ then go to Step $k.l$+1 with $\mu'''_{k,l} = \mu'''_{k,l-1}$. If $m_k^l \subseteq \mu'''_{k,l-1}$ and $l = L_k$ then go to Step k.End with $\mu'''_{k,L_k} = \mu'''_{k,l-1}$. If $m_k^l \not\subseteq \mu'''_{k,l-1}$ then $\mu'_{k,l} = \mu'''_{k,l-1} - \{(i, \mu'''_{k,l-1}(i)) \mid (i, \mu^T(i)) \in m_k^l \text{ and } \mu'''_{k,l-1}(i) \neq i\} + \{(i,s) \mid i,s \in c_k^l \text{ and } s \mapsto i\} - \{(j,s) \in \mu'''_{k,l-1} \mid s \in c_k^l,\ \mu'''_{k,l-1}(s) \cap c_k^l = \emptyset,\ \#\mu'''_{k,l-1}(s) = q_s \text{ and } F_s(j) > F_s(j') \text{ for all } j' \in \mu'''_{k,l-1}(s), j' \neq j\}$. Starting from $\mu'''_{k,l-1}$, looking forward towards μ^T, the coalition of students belonging to c_k^l has now incentives to deviate to $\mu'_{k,l}$ where each student in c_k^l is assigned to the school where she has the highest priority among students belonging to $I \setminus (\bigcup_{k'=1}^{k-1} I_{k'})$. Given that the matches $\bigcup_{k'=1}^{k-1} M_{k'} \subseteq \mu'''_{k-1,L_{k-1}}$ are already settled and remain fixed, students belonging to c_k^l obtain in μ^T their best possible match. Schools have incentives to accept those students because either they do not have full capacity or the new student replaces the student who had the lowest priority among the students enrolled at the school. Next, students belonging to c_k^l leave their school to reach $\mu''_{k,l} = \mu'_{k,l} - \{(i,s) \mid i,s \in c_k^l \text{ and } s \mapsto i\}$. Next, each student belonging to c_k^l joins her most preferred school (constrained to $\bigcup_{k'=1}^{k-1} M_{k'}$ being fixed) to reach $\mu'''_{k,l} = \mu''_{k,l} + \{(i,s) \mid i,s \in c_k^l \text{ and } i \mapsto s\}$. Schools accept those students since they have (at least) one vacant position. We reach $\mu'''_{k,l}$ with $m_k^l \subseteq \mu'''_{k,l}$ and so students belonging to c_k^l are assigned to the same school as in μ^T. If $l \neq L_k$, then go to Step $k.l$+1. Otherwise, go to Step k.End with $\mu'''_{k,L_k} = \mu'''_{k,l}$.

Step k.End. We have reached μ'''_{k,L_k} with $\bigcup_{k'=1}^{k} M_{k'} \subseteq \mu'''_{k,L_k}$. If $\mu'''_{k,L_k} = \mu^T$ then the process ends. Otherwise, go to Step k+1.

End. The process goes on until Step $\bar{k}$ where we reach $\mu'''_{\bar{k},L_{\bar{k}}} = \bigcup_{k=1}^{\bar{k}} M_k = \mu^T$.

□

The matching obtained from the TTC algorithm is always Pareto efficient but may not be stable when students are myopic. Theorem 1 shows that, once students are farsighted, the matching obtained from the TTC algorithm becomes stable.[12] By means of Example 1 we provide the basic intuition behind Theorem 1 and its proof.

Example 1 (Haeringer, 2017)**.** Consider a school choice problem $\langle I, S, q, P, F\rangle$ with $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2, s_3\}$. Students' preferences and schools' priorities and capacities are as follows.

Students			
P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_1	s_1	s_2	s_1
s_2	s_2	s_1	s_3
s_3	s_3	s_3	s_2

Schools			
	F_{s_1}	F_{s_2}	F_{s_3}
q_s	2	1	1
	i_1	i_1	i_2
	i_3	i_2	i_3
	i_4	i_4	i_4
	i_2	i_3	i_1

In Example 1, $\mu^T = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_3)\}$ is the matching obtained from the TTC algorithm. In the first round of the TTC algorithm, there is one cycle where student i_1 points to school s_1 and school s_1 points to student i_1. That is, $C_1 = \{c_1^1\}$ with $c_1^1 = \{s_1, i_1\}$. Student i_1 is matched to school s_1: $m_1^1 = \{(i_1, s_1)\}$ and school s_1 has only one leftover seat. In the second round of the TTC algorithm, there is one cycle where student i_2 points to school s_1, school s_1 points to student i_3, student i_3 points to school s_2 and school s_2 points to student i_2. That is, $C_2 = \{c_2^1\}$ with $c_2^1 = \{s_1, i_3, s_2, i_2\}$. Student i_2 is matched to school s_1 and student i_3 is matched to school s_2: $m_2^1 = \{(i_2, s_1), (i_3, s_2)\}$, and so i_2 and i_3 exchange their priority. In the

[12]This result is robust to the incorporation of various forms of maximality in the definition of farsighted improving path, like the strong rational expectations farsighted stable set in Dutta and Vohra (2017) and absolute maximality as in Ray and Vohra (2019). See also Herings, Mauleon and Vannetelbosch (2020).

third round of the TTC algorithm, there is only one leftover student, i_4, who points to school s_3 and school s_3 points to student i_4. That is, $C_3 = \{c_3^1\}$ with $c_3^1 = \{s_3, i_4\}$. Student i_4 is matched to school s_3: $m_3^1 = \{(i_4, s_3)\}$, and so $\mu^T = m_1^1 \cup m_2^1 \cup m_3^1$.

From Theorem 1 we know that $\{\mu^T\}$ is a farsighted stable set. Indeed, from any $\mu \neq \mu^T$ there exists a farsighted improving path leading to μ^T. We now provide the basic mechanism behind the construction of a farsighted improving path leading to μ^T. Take for instance the matching $\mu_0 = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_1)\}$. We now construct a farsighted improving from μ_0 to $\mu^T = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_3)\} = \mu_4$ following the steps as in the proof of Theorem 1. First, we consider students and schools belonging to the cycles in C_1. Since $m_1^1 = \{(i_1, s_1)\} \subseteq \mu_0$, student i_1 stays matched to school s_1 along the farsighted improving path, i.e. $m_1^1 = \{(i_1, s_1)\} \subseteq \mu_l$, $0 \leq l \leq 4$. Next, we consider students and schools belonging to the cycles in C_2. Notice that $m_2^1 = \{(i_2, s_1), (i_3, s_2)\} \cap \mu_0 = \emptyset$. Looking forward towards μ^T, the coalition $N_0 = \{i_2, i_3, s_1, s_2\}$ deviates so that student i_3 joins school s_1 and student i_2 joins schools s_2 to reach the matching $\mu_1 = \{(i_1, s_1), (i_2, s_2), (i_3, s_1), (i_4, i_4)\}$ where students i_2 and i_3 are matched to the schools where they have priority. By doing so, they push student i_4 out of school s_1. Next, the coalition $N_1 = \{i_2, i_3\}$ deviates so that students i_2 and i_3 leave, respectively, schools s_2 and s_1 to reach the matching $\mu_2 = \{(i_1, s_1), (i_2, i_2), (i_3, i_3), (i_4, i_4)\}$ where both students are not assigned to any school. They are temporarily worse off, but they anticipate to end up in μ^T. Next, the coalition $N_2 = \{i_2, i_3, s_1, s_2\}$ deviates so that student i_2 joins school s_1 and student i_3 joins schools s_2 to reach the matching $\mu_3 = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, i_4)\}$ with $m_2^1 = \{(i_2, s_1), (i_3, s_2)\} \subseteq \mu_3$. Both schools accept to enroll those students because they are not at full capacity. Finally, we consider students and schools belonging to the cycles in C_3. Since $m_3^1 = \{(i_4, s_3)\} \cap \mu_3 = \emptyset$, the coalition $N_3 = \{i_4, s_3\}$ deviates so that student i_4 joins school s_3 to form the match (i_4, s_3) and to reach the matching $\mu_4 = \mu^T$. Thus, $\mu^T \in \phi(\mu_0)$.

In Example 1, $\mu^D = \{(i_1, s_1), (i_2, s_2), (i_3, s_1), (i_4, s_3)\}$ is the matching obtained from the Deferred Acceptance (DA) algorithm, $\mu^B = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_1)\}$ is the matching obtained from the Immediate Acceptance (IA) algorithm (i.e. the Boston mechanism). Thus, $\mu^T \neq \mu^D \neq \mu^B$. We next show that in Example 1, once students are farsighted, the matching obtained from the DA algorithm and the matching obtained from the IA algorithm are unstable.

Since students are at least as well off and some of them (i_2 and i_3) are strictly better off in μ^T than in μ^D, we have that there is no farsighted improving path from μ^T to μ^D. That is, $\mu^D \notin \phi(\mu^T)$. Hence, $\{\mu^D\}$ is not a farsighted stable set since (ES) is violated. Let

$$\begin{aligned}
\mu^1 &= \{(i_1, s_1), (i_2, i_2), (i_3, s_2), (i_4, s_1)\}, \\
\mu^2 &= \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_1)\} = \mu^B, \\
\mu^3 &= \{(i_1, s_1), (i_2, s_2), (i_3, i_3), (i_4, s_1)\}, \\
\mu^4 &= \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_1)\}, \\
\mu^5 &= \{(i_1, s_1), (i_2, s_2), (i_3, s_1), (i_4, i_4)\}.
\end{aligned}$$

Computing the farsighted improving paths emanating from μ^T, we get $\phi(\mu^T) = \{\mu^1, \mu^2, \mu^3, \mu^4\}$. Notice that $\mu^5 \notin \phi(\mu^T)$ since student i_4 is worse off in μ^5 than in μ^T. From μ^1, μ^2, μ^3, μ^4 and μ^5, there is a farsighted improving to μ^D. That is, $\mu^D \in \phi(\mu)$ for $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$. From μ^D there is only a farsighted improving path to μ^T; i.e. $\phi(\mu^D) = \{\mu^T\}$. For a set $V \supseteq \{\mu^D\}$ to be a farsighted stable set, we need that (i) $\mu^T \notin V$ (otherwise (IS) is violated), (ii) a single $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4\}$ should belong to V to satisfy (ES) since $\mu^D \notin \phi(\mu^T)$. But, V would then violate (IS) since $\mu^D \in \phi(\mu)$ for $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$. Thus, there is no V such that $\mu^D \in V$ that is a farsighted stable set in Example 1.

Since $\phi(\mu^D) = \{\mu^T\}$, there is no farsighted improving path from μ^D to μ^B. Thus, $V = \{\mu^B\}$ does not satisfy (ES), and hence $V = \{\mu^B\}$ is not a farsighted stable set. Moreover, a set $V \supseteq \{\mu^B, \mu^D\}$ cannot be a farsighted stable since $\mu^D \in \phi(\mu^B)$. Otherwise, V would violate (IS) since there is a farsighted stable improving path from μ^B to μ^D.

Is $V = \{\mu^T\}$ the unique farsighted stable set in Example 1? Any other set V' such that $\mu^T \in V'$ violates (IS), and hence $\mu^T \notin V'$. Then, $\mu^D \in V'$ because, otherwise, V' violates (ES) since $\phi(\mu^D) = \{\mu^T\}$. However, as shown before, there is no V' such that $\mu^D \in V'$ that is a farsighted stable set. Thus, we have that $V = \{\mu^T\}$ is the unique farsighted stable set.

Remark 1. There are school choice problems such that the matching obtained from the Deferred Acceptance (DA) algorithm does not belong to any farsightedly stable set.

Since the matching obtained from the IA algorithm is Pareto efficient, Example 1 also shows that there are school choice problems where some Pareto efficient matching does not belong to any farsighted stable set. Thus, Pareto efficiency is not a sufficient condition for guaranteeing the stability of a matching when students are farsighted.

Remark 2. There are school choice problems such that some Pareto efficient matching does not belong to any farsighted stable set.

Corollary 1. *Let* $\langle I, S, q, P, F \rangle$ *be a school choice problem and* μ^T *be the matching obtained from the Top Trading Cycles mechanism after* $\bar{k}$ *steps. From any* $\mu \neq \mu^T$ *there is a farsighted improving path to* μ^T *with* $\mu_0 = \mu$ *and* $\mu_L = \mu^T$ *such that for every* $l \in \{0, \ldots, L-1\}$ *there is a coalition* $N_l \subseteq \bigcup_{k=1}^{\bar{k}} C_k$ *that enforces* μ_{l+1} *from* μ_l*.*

Corollary 1 follows from the proof of Theorem 1. Notice that $\bigcup_{k=1}^{\bar{k}} C_k$ is simply the collection of sets where each element is a set consisting of students and schools belonging to a cycle obtained from the TTC algorithm. Definition 2 of a farsighted improving path is quite permissive in terms of the size of the coalition N_l that enforces μ_{l+1} from μ_l. However, Corollary 1 tells us that there exists a farsighted improving path from $\mu \neq \mu^T$ to μ^T with $\mu_0 = \mu$ and $\mu_L = \mu^T$ such that for every $l \in \{0, \ldots, L-1\}$ the coalition N_l that enforces μ_{l+1} from μ_l consists of students (and possibly schools) who are part of the same cycle in the TTC algorithm. Thus, for getting Theorem 1, it is sufficient to allow a deviating coalition (involving more than one student) to be composed exclusively of students (and possibly their schools) who are exchanging their priorities among themselves in the TTC algorithm. Such restriction seems not too demanding since students who coordinate their moves are the ones who exchange their priorities.

5 Limited Farsightedness

How much farsightedness from the students do we need to stabilize the matching obtained from the TTC algorithm? To answer this question we propose the notion of horizon-k farsighted stable set for school choice problems to study the matchings that are stable when students are limited in their degree of farsightedness. A horizon-k farsighted improving path for school choice problems is a sequence of matchings that can emerge when limited farsighted students form or destroy matches based on the improvement the k-steps ahead matching offers them relative to the current one while myopic schools form or destroy matches based on the improvement the next matching in the sequence offers them relative to the current one. A set of matchings is a horizon-k farsighted stable set if (IS) for any two matchings belonging to the set, there is no horizon-k farsighted improving path connecting from one matching to the other one, and (ES) there always exists a horizon-k farsighted improving path from every matching outside the set to some matching within the set.

Definition 4. Let $\langle I, S, q, P, F\rangle$ be a school choice problem. A horizon-k farsighted improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, \ldots, L-1\}$ there is a coalition $N_l \subseteq I \cup S$ that can enforce μ_{l+1} from μ_l and

(i) $\mu_{\min\{l+k,L\}}(i) R_i \mu_l(i)$ for all $i \in N_l \cap I$ and $\mu_{\min\{l+k,L\}}(j) P_j \mu_l(j)$ for some $j \in N_l \cap I$,

(ii) For every $s \in N_l \cap S$ such that $\#\mu_l(s) + \#\{i \in I \mid i \notin \mu_l(s), i \in \mu_{l+1}(s)\} > q_s$, there is $\{i_1, \ldots, i_J\} \subseteq \{i \in I \mid i \notin \mu_l(s), i \in \mu_{l+1}(s)\}$ and $\{j_1, \ldots, j_J\} = \{i \in I \mid i \in \mu_l(s), i \notin \mu_{l+1}(s)\}$ such that

$$
\begin{aligned}
F_s(i_1) &< F_s(j_1) \\
F_s(i_2) &< F_s(j_2) \\
&\vdots \\
F_s(i_J) &< F_s(j_J).
\end{aligned}
$$

Definition 4 tells us that a horizon-k farsighted improving path for school choice problems consists of a sequence of matchings where along the sequence students form or destroy matches based on the improvement the k-steps ahead matching offers them relative to the current one. Precisely, along a horizon-k farsighted improving path, each time some student i is on the move she is comparing her current match (i.e. $\mu_l(i)$) with the match she will get k-steps ahead on the sequence (i.e. $\mu_{l+k}(i)$) except if the end matching of the sequence lies within her horizon (i.e. $L < l+k$). In such a case, she simply compares her current match (i.e. $\mu_l(i)$) with the end match (i.e. μ_L). Schools continue to accept any student on their priority lists unless they have full capacity. In the case of full capacity, a school $s \in N_l \cap S$ accepts to replace the match μ_l by μ_{l+1} if each student $i \in \{j \in I \mid j \in \mu_l(s), j \notin \mu_{l+1}(s)\}$ who leaves or is evicted from school s from μ_l to μ_{l+1} is replaced by a newly enrolled student who has a higher priority.

Let some $\mu \in \mathcal{M}$ be given. If there exists a horizon-k farsighted improving path from a matching μ to a matching μ', then we write $\mu \rightarrow_k \mu'$. The set of matchings $\mu' \in \mathcal{M}$ such that there is a horizon-k farsighted improving path from μ to μ' is denoted by $\phi_k(\mu)$, so $\phi_k(\mu) = \{\mu' \in \mathcal{M} \mid \mu \rightarrow_k \mu'\}$.

Definition 5. Let $\langle I, S, q, P, F\rangle$ be a school choice problem. A set of matchings $V \subseteq \mathcal{M}$ is a horizon-k farsighted stable set if it satisfies:

(i) For every $\mu, \mu' \in V$, it holds that $\mu' \notin \phi_k(\mu)$.

(ii) For every $\mu \in \mathcal{M} \setminus V$, it holds that $\phi_k(\mu) \cap V \neq \emptyset$.

From the construction of a farsighted improving path in the proof of Theorem 1 we have that students belonging to a cycle only need to look forward three steps ahead to have incentives for engaging a move towards the matches they have in the matching obtained from the TTC algorithm, μ^T. Once they reach those matches they do not move afterwards. The three steps consist of (i) getting first a seat at the school they have priority, (ii) leaving that school and by doing so, guaranteeing a free seat at that school, (iii) joining the school they match to in μ^T. Hence, for $k \geq 3$, there exists a horizon-k farsighted improving from any $\mu \neq \mu^T$ to μ^T, and so $\{\mu^T\}$ is a horizon-k farsighted stable set.[13]

Corollary 2. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem and μ^T be the matching obtained from the Top Trading Cycles mechanism. The singleton set $\{\mu^T\}$ is a horizon-k farsighted stable set for $k \geq 3$.*

6 Variations of The TTC Algorithm

6.1 Equitable Top Trading Cycles Algorithm

Hakimov and Kesten (2018) introduce the Equitable Top Trading Cycles mechanism for selecting a matching for each school problem by means of the Equitable

[13]In Example 1, it is sufficient for the students who belong to c_2^1 to look forward towards μ_3 when they participate to the moves from μ_0 to μ_4. Indeed, they are not affected by the move from μ_3 to μ_4 since they remain with the same matches.

Top Trading Cycles algorithm (ETTC). They show that the ETTC mechanism is Pareto-efficient and group strategy-proof and eliminates more avoidable justified envy situations than the TTC. Instead of allowing only the current highest priority students to participate in the trading process, the ETTC assigns all slots of each school s to all the q_s students with the highest priorities in each school, giving one slot to each student and endowing them with equal trading power. The terms of trade are next determined by a pointing rule specifying for each student-school pair which student-school pair should be pointed to among those who contain the remaining favourite school.

In the ETTC, each student-school pair (i, s) points to the student-school pair (i', s') containing the highest priority student for s, the school contained in the former pair. By doing so, the ETTC mechanism ensures that the students involved in a cycle of student-school pairs have the highest priority for their favourite schools among their competitors at that step of the trading market.

Example 2. Consider a school choice problem $\langle I, S, q, P, F \rangle$ with eight students, $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$ and four schools, $S = \{s_1, s_2, s_3, s_4\}$. Students' preferences and schools' priorities and capacities are as follows.

Students

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}
s_2	s_3	s_1	s_1	s_4	s_2	s_4	s_1
s_1	s_1	s_3	s_3	s_2	s_4	s_2	s_3
s_3	s_2	s_4	s_2	s_3	s_3	s_1	s_2
s_4	s_4	s_2	s_4	s_1	s_1	s_3	s_4

Schools

	F_{s_1}	F_{s_2}	F_{s_3}	F_{s_4}
q_s	1	3	3	1
	i_1	i_4	i_1	i_2
	i_2	i_2	i_5	i_6
	i_5	i_3	i_3	i_5
	i_7	i_8	i_7	i_3
	i_6	i_7	i_4	i_7
	i_3	i_1	i_2	i_8
	i_8	i_5	i_6	i_4
	i_4	i_6	i_8	i_1

Let μ^E be the matching obtained from the ETTC mechanism. We illustrate the mechanism behind the ETTC algorithm by means of Example 2 which is adapted from Hakimov and Kesten (2018). A formal description of the ETTC algorithm can be found in Appendix A.1.

In the inheritance round of the first step of the ETTC algorithm, all seats are available to inherit and so, students are assigned to seats according to the priority orders F to form the following student-school pairs: (i_1, s_1), (i_4, s_2), (i_2, s_2), (i_3, s_2), (i_1, s_3), (i_5, s_3), (i_3, s_3) and (i_2, s_4). Next, each student-school pair (i, s) points to the student-pair (i', s') such that s' is the top choice of student i and i' has the highest priority for school s among the students who are assigned a seat at school s' in the inheritance round. That is, (i_1, s_1) points to (i_2, s_2), (i_2, s_2) points to (i_3, s_3), (i_3, s_3) points to (i_1, s_1). In addition, (i_5, s_3) points to (i_2, s_4) and (i_2, s_4) points to (i_5, s_3). Finally, (i_4, s_2) points to (i_1, s_1). Hence, there are two cycles: $(i_1, s_1) \mapsto (i_2, s_2) \mapsto (i_3, s_3) \mapsto (i_1, s_1)$ and $(i_5, s_3) \mapsto (i_2, s_4) \mapsto (i_5, s_3)$ with student i_2 participating to both cycles.[14] It leads to the following matches: (i_1, s_2), (i_2, s_3), (i_3, s_1) and (i_5, s_4).

There is one seat at school s_2 and one seat at school s_3 to be inherited from the first step. Since there is student i_4 who was assigned a seat at school s_2 in the inheritance round of the first step and i_4 was not matched to some school in step one, there is no inheritance of the s_2 seat at this step. Since all students who are assigned a seat at school s_3 in the inheritance round of the first step are matched to some school in step one, the remaining seats of school s_3 are inherited by students i_7 and i_4. Thus, the student-school pairs in the inheritance round of the second step are (i_4, s_2), (i_7, s_3) and (i_4, s_3). Since student i_4 is already assigned one seat from her best choice school (i.e. s_3), then all student-school pairs containing her point to that student-school pair (i.e. $(i_4, s_2) \mapsto (i_4, s_3)$) and the student-school pair (i_4, s_3) points to (i_4, s_3) and the match (i_4, s_3) is formed.

Since there is student i_7 who was assigned a seat at school s_3 in the inheritance round of the second step and i_7 was not matched to some school in step two, there is no inheritance of the s_3 seat at this step. Since the student who is assigned a seat at school s_2 in the inheritance round of the second step is matched to some school in step two, the remaining seats of school s_2 are inherited by students i_8 and i_7. Thus, the student-school pairs in the inheritance round of the third step are (i_7, s_3), (i_8, s_2) and (i_7, s_2). Since student i_7 is already assigned one seat from her best choice school (i.e. s_2), then all student-school pairs containing her point to that student-school pair (i.e. $(i_7, s_3) \mapsto (i_7, s_2)$) and the student-school pair (i_7, s_2) points to (i_7, s_2) and the match (i_7, s_2) is formed.

[14] There is always at least one cycle in each step. If some student appears in the same cycle or in different cycles with different schools, then she is definitely assigned a seat at her top choice among those schools while the other seats she was pointing to remain to be inherited later on.

Since there is student i_8 who was assigned a seat at school s_2 in the inheritance round of the third step and i_8 was not matched to some school in step three, there is no inheritance of the s_2 seat. Since the only student who is assigned a seat at school s_3 in the inheritance round of the third step is matched to some school in step three, the remaining seat of school s_3 is inherited by student i_6. Thus, the student-school pairs in the inheritance round of the fourth step are (i_8, s_2) and (i_6, s_3). Then, (i_8, s_2) points to (i_6, s_3) and (i_6, s_3) points to (i_8, s_2). So, there is one cycle $(i_8, s_2) \mapsto (i_6, s_3) \mapsto (i_8, s_2)$ leading to the following matches: (i_8, s_3) and (i_6, s_2). We reach the matching obtained from the ETTC algorithm, $\mu^E = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_3), (i_5, s_4), (i_6, s_2), (i_7, s_2), (i_8, s_3)\}$. Notice that the matching obtained from the ETTC algorithm differs from the TTC matching, $\mu^T = \{(i_1, s_2), (i_2, s_3), (i_3, s_3), (i_4, s_1), (i_5, s_4), (i_6, s_2), (i_7, s_2), (i_8, s_3)\}$.

Theorem 2. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem and μ^E be the matching obtained from the Equitable Top Trading Cycles mechanism. The singleton set $\{\mu^E\}$ is a farsighted stable set.*

The proof of Theorem 2 can be found in Appendix A.1. Since $\{\mu^E\}$ is a singleton set, internal stability (IS) is satisfied. We provide the intuition for external stability (ES) by means of Example 2. Take for instance the matching $\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_3), (i_4, s_3), (i_5, s_2), (i_6, s_4), (i_7, s_2), (i_8, s_2)\}$. Following the steps as in the proof of Theorem 2, we construct a farsighted improving from μ_0 to $\mu^E = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_3), (i_5, s_4), (i_6, s_2), (i_7, s_2), (i_8, s_3)\} = \mu_7$. In the first round of ETTC, there are two cycles, one cycle $c_1^1 = ((i_1, s_1), (i_2, s_2), (i_3, s_3))$ and a second cycle $c_1^2 = ((i_5, s_3), (i_2, s_4))$. Notice that student i_2 is involved in both cycles. Looking forward towards μ^E, the coalition of students and schools belonging to c_1^1, i.e. $\{i_1, i_2, i_3, s_1, s_2, s_3\}$, deviates from μ_0 to reach the matching $\mu_1 = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_3), (i_5, i_5), (i_6, s_4), (i_7, s_2), (i_8, s_2)\}$ so that student i_1 joins school s_1, student i_2 joins school s_2 and student i_3 remains with school s_3. That is, each student is matched to the school from the pair student-school in c_1^1 she belongs to. By doing so, student i_1 already vacates a slot at school s_3. Next, the coalition of students $\{i_1, i_2, i_3\}$ deviates from μ_1 so that student i_1 leaves school s_1, student i_2 leaves school s_2 and student i_3 leaves school s_3 to reach the matching $\mu_2 = \{(i_1, i_1), (i_2, i_2), (i_3, i_3), (i_4, s_3), (i_5, i_5), (i_6, s_4), (i_7, s_2), (i_8, s_2)\}$ where all three students are unmatched. Next, the coalition of students and schools belonging to c_1^2, i.e. $\{i_2, i_5, s_3, s_4\}$, deviates from μ_2 to reach the matching $\mu_3 =$

$\{(i_1, i_1), (i_2, s_4), (i_3, i_3), (i_4, s_3), (i_5, s_3), (i_6, i_6), (i_7, s_2), (i_8, s_2)\}$ so that student i_5 joins school s_3 and student i_2 joins school s_4. Next, the coalition of students $\{i_2, i_5\}$ deviates from μ_3 so that student i_2 leaves school s_4 and student i_5 leaves school s_3 to reach the matching $\mu_4 = \{(i_1, i_1), (i_2, i_2), (i_3, i_3), (i_4, s_3), (i_5, i_5), (i_6, i_6), (i_7, s_2), (i_8, s_2)\}$ so that both students are unmatched. Next, the students $\{i_1, i_2, i_3, i_5\}$ join their schools in μ^E leading to $\{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_3), (i_5, s_4), (i_6, i_6), (i_7, i_7), (i_8, s_2)\} = \mu_5$.

Notice that, in the second step, student i_4 is already matched to s_3 in μ_5 and $c_2^1 = ((i_4, s_3))$. Hence, she simply remains matched to $s_3 = \mu^E(i_4)$ along the farsighted improving path. In the third step, given that $c_3^1 = ((i_7, s_2))$, student i_7 remains matched to school $s_2 = \mu^E(i_7)$. In the fourth step, there is only one cycle c_4^1 involving student-school pairs $(i_6, s_3), (i_8, s_2)$. Since student i_8 is already matched to s_2 and student i_6 is unmatched in μ_6, only student i_8 leaves her school s_2 to reach $\{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_3), (i_5, s_4), (i_6, i_6), (i_7, s_2), (i_8, i_8)\} = \mu_6$ where both are unmatched. Finally, student i_6 joins school s_2 and student i_8 joins school s_3 to reach the ETTC matching $\mu^E = \mu_7$.

Thus, $\mu^E \in \phi(\mu_0)$. Again, it holds in general that, from any $\mu \neq \mu^E$ there exists a farsighted improving path leading to μ^E. Thus, the matching obtained from the ETTC algorithm also preserves the property of being stable once students are farsighted.

How much farsightedness from the students do we need to stabilize the matching obtained from the ETTC algorithm? Let $C_k = \{c_k^1, c_k^2, \ldots, c_k^{L_k}\}$ be the set of cycles in Step k.B of the ETTC algorithm (formally described in Appendix A.1), for $k = 1, \ldots, \bar{k}$. Let $\tau(i, c_k^l) = \#\{(j, s) \in c_k^l \mid j = i\}$ be the number of distinct pairs involving student i in cycle c_k^l and let $\tau(c_k^l) = \max_{i \in \{j \in I | (j,s) \in c_k^l\}} \tau(i, c_k^l)$. From the construction of a farsighted improving path in the proof of Theorem 2, we have that the minimum degree of farsightedness that guarantees that students involve in any cycle always reach their ETTC assignment is

$$\tau^{\max} = \max_{k \in \{1, \ldots, \bar{k}\}} \left(\sum_{l=1}^{L_k} 2\tau(c_k^l) \right) + 1.$$

Corollary 3. *Let $\langle I, S, q, P, F \rangle$ be a school choice problem and μ^E be the matching obtained from the Equitable Top Trading Cycles mechanism. The singleton set $\{\mu^E\}$ is a horizon-k farsighted stable set for $k \geq \tau^{max}$.*

Notice that in Example 2, the farsighted improving path from μ_0 to μ^E requires a degree of farsightedness equal to 5 so that students i_1, i_2 and i_3 have incentives to engage a move in μ_0 looking forward towards μ_5 where they already obtain their ETTC assignment. Since student i_2 is involved in two cycles in the first round of ETTC, $\tau^{\max}$ is equal to 5. Thus, compared to the TTC, the ETTC improves in terms of no justified envy, but requires more farsightedness on behalf of students.

6.2 (First) Clinch and Trade Algorithm

Morrill (2015) introduces two variations of the Top Trading Cycles mechanism for selecting a matching for each school problem: the First Clinch and Trade mechanism (FCT) and the Clinch and Trade mechanism (CT). Both mechanisms intend to mitigate the following problem. In the TTC mechanism, if a student i's most preferred school is s and the student has one of the q_s highest priorities at s, then i is always assigned to s. However, until i has the highest priority at s, the TTC mechanism allows i to trade her priority at other schools to be assigned to s. Such trade may cause distortions regarding the elimination of justified envy.

In the First Clinch and Trade algorithm (FCT), a student that initially has one of the q_s highest priorities at a school s (she is guaranteed a seat at s), cannot trade with another student to get s. The FCT algorithm runs basically the TTC algorithm but, at each round, if a student points at a school where she is guaranteed a seat, the student is assigned to the school and cannot trade her priority. For the remaining students, the TTC is run and the students who have the highest priorities at some schools are allowed to trade their priorities and are assigned their top choices.[15]

Example 3 (Morrill, 2015)**.** Consider a school choice problem $\langle I, S, q, P, F\rangle$ with $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2\}$. Students' preferences and schools' priorities and capacities are as follows.

Students		
P_{i_1}	P_{i_2}	P_{i_3}
s_2	s_1	s_2
s_1	s_2	s_1

	Schools	
	F_{s_1}	F_{s_2}
q_s	2	1
	i_1	i_2
	i_2	i_3
	i_3	i_1

[15]Morrill (2015) shows that the FCT is Pareto efficient, strategy-proof, non-bossy, group strategy-proof, reallocation proof and independent of the order in which cycles are processed.

Let μ^F be the matching obtained from the FCT mechanism. A formal description of the FCT algorithm can be found in Morrill (2015) and Atay, Mauleon and Vannetelbosch (2022a). By means of Example 3 we illustrate the mechanism behind the FCT algorithm. In the first round, each student points to her top choice school. That is, i_1 points to s_2, i_2 points to s_1 and i_3 points to s_2. Student i_1 and student i_2 are guaranteed admissions to school s_1 since both have one of the two highest rankings at school s_1. Student i_2 is also guaranteed admission to school s_2 since she is ranked first at school s_2. Student i_2 is pointing to s_1, and so she is clinched to school s_1 and the match (i_2, s_1) is formed. Student i_1 is not pointing to a school where she is guaranteed admission. Hence, she is not clinched to any school and she participates next with i_3 to the trading procedure. Schools s_1 and s_2 point to their highest ranked student, respectively i_1 and i_2. Hence, there is no cycle and no match is formed. In the second round, each student points to her top choice school that has still available capacity. That is, i_1 points to s_2 and i_3 points to s_2. Guaranteed admissions do not change. Hence, nor i_1 nor i_3 are clinched to some school and so they participate next to the trading procedure. School s_1 points to i_1 while school s_2 points now to i_3 and so the match (i_3, s_2) is formed. Student i_1 remains unmatched. In the third round, each remaining student points to her preferred school that has still available capacity. That is, student i_1 points now to school s_1. Since she is guaranteed admission to school s_1, she is clinched and assigned to school s_1. We obtain the matching $\mu^F = \{(i_1, s_1), (i_2, s_1), (i_3, s_2)\}$.

The matching obtained from the FCT algorithm differs from the matching obtained from the TTC algorithm, $\mu^T = \{(i_1, s_2), (i_2, s_1), (i_3, s_1)\}$. In the TTC mechanism, students i_1 and i_2 first exchange their priorities to form the matches (i_1, s_2) and (i_2, s_1). Student i_1 has priority at s_1 while student i_2 has priority at s_2. However, student i_3 is ranked above student i_1 at school s_2. In addition, student i_2 is guaranteed admission at school s_1. The FCT mechanism intends to remedy to such drawback.

While the FCT algorithm does not update the students that are able to clinch her most preferred school, the Clinch and Trade algorithm (CT) removes from the priority list of each school the students that are guaranteed a seat. As a result, the priorities of the remaining students weakly improve and thus, students that initially are not guaranteed a seat at their most preferred school may now be guaranteed one of the remaining seats. Let μ^C be the matching obtained from the CT mechanism.

A formal description of the CT algorithm can be found in Morrill (2015) and Atay, Mauleon and Vannetelbosch (2022a).[16] In Example 3 both mechanisms lead to the same matching.[17]

Theorem 3. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem. Let μ^F and μ^C be the matchings obtained from the First Clinch and Trade mechanism and the Clinch and Trade mechanism, respectively. Both singleton sets $\{\mu^F\}$ and $\{\mu^C\}$ are farsighted stable sets.*

The proof is similar to the one for Theorem 1 and can be found in Atay, Mauleon and Vannetelbosch (2022a). Since $\{\mu^F\}$ and $\{\mu^C\}$ are singleton sets, internal stability (IS) is satisfied.

We now provide the intuition for external stability (ES) of μ^F by means of Example 3. Take for instance the matching $\mu_0 = \{(i_1, s_2), (i_2, s_1), (i_3, s_1)\} = \mu^T$. We now construct a farsighted improving from μ_0 to $\mu^F = \{(i_1, s_1), (i_2, s_1), (i_3, s_2)\} = \mu_2$. First, we consider student i_2 who is the only student to be clinched in the first round of the FCT. Since student i_2 is matched to school s_1 in both μ_0 and μ_F she does not participate to any deviation and remains clinched to s_1 along the farsighted improving path. That is, $(i_2, s_1) \in \mu_l$, $l = 0, 1, 2$. There is no cycle between schools and students who are not clinched in the first round of the FCT. In the second round of the FCT, none of the remaining students is clinched to some school. However, student i_3 and school s_2 form a cycle. So, looking forward towards μ^T, the coalition $N_0 = \{i_3, s_2\}$ deviates from μ_0 so that student i_3 joins school s_2 to reach the matching $\mu_1 = \{(i_1, i_1), (i_2, s_1), (i_3, s_2)\}$. Student i_3 is matched to her preferred school s_2 in μ_1 and she has a higher priority than i_1 at s_2. By doing so, student i_1 is pushed out of school s_2. In the third round of the FCT, student i_1 points to school s_1 and is guaranteed admission at school s_1. So, from μ_1, the coalition $N_1 = \{i_1, s_1\}$ deviates so that student i_1 joins school s_1 to reach the matching $\mu_2 = \{(i_1, s_1), (i_2, s_1), (i_3, s_2)\} = \mu^F$. Thus, $\mu^F \in \phi(\mu^T)$.

In fact, it holds in general that, from any $\mu \neq \mu^F$ there exists a farsighted improving path leading to μ^F. So, $\{\mu^F\}$ is a farsighted stable set and the match-

[16] Morrill (2015) shows that the CT mechanism is Pareto efficient and strategy-proof. Unlike the TTC mechanism, the CT mechanism is bossy, not group strategy-proof, and not independent of the order in which cycles are processed.

[17] In Example 1, $\mu^T = \mu^F = \mu^C = \mu^E \neq \mu^D$. In Example 2, $\mu^E \neq \mu^T = \mu^F = \mu^C \neq \mu^D$. In Example 3, $\mu^T = \mu^E \neq \mu^F = \mu^C = \mu^D$.

ing obtained from the FCT algorithm preserves the property of being stable once students are farsighted. Likewise, $\{\mu^C\}$ is a farsighted stable set.

Similarly to the TTC, we have that students belonging to a cycle only need to look forward three steps ahead to have incentives for engaging a move towards the matches they have in the matching obtained from the FCT (CT) algorithm, μ^F (μ^C).

Corollary 4. *Let $\langle I, S, q, P, F\rangle$ be a school choice problem. Let μ^F and μ^C be the matchings obtained from the First Clinch and Trade mechanism and the Clinch and Trade mechanism, respectively. Both singleton sets $\{\mu^F\}$ and $\{\mu^C\}$ are horizon-k farsighted stable set for $k \geq 3$.*

7 Conclusion

We have considered priority-based school choice problems. Once students are farsighted, the matching obtained from the TTC mechanism becomes stable: a singleton set consisting of the TTC matching is a farsighted stable set. However, the matching obtained from the DA mechanism may not belong to any farsighted stable set. Hence, the TTC mechanism provides an assignment that is not only Pareto efficient but also farsightedly stable. Moreover, looking forward three steps ahead is already sufficient for stabilizing the matching obtained from the TTC. Since the choice between the DA mechanism or the TTC mechanism usually depends on the priorities of the policy makers, farsightedness and Pareto efficiency may tip the balance in favor of TTC or one of its variations.

In this paper, we have focused on priority-based school choice problems where arbitrary groups of students (and schools) can move together and where schools could have more than one seat of capacity. We have shown that the TTC matching is a farsighted stable set when allowing deviations of coalitions of arbitrary size. In matching theory, it is often assumed that only individual or pairwise deviations are feasible and most results are robust to deviations of coalitions of arbitrary size. However, this is not always true once agents are farsighted. In a follow-up paper, Atay, Mauleon and Vannetelbosch (2022b) have found that, for one-to-one priority-based matching problems (i.e., problems with only one vacant seat (object) to be assigned among the set of agents) where only individual or pairwise deviations are allowed, the TTC matching is stable provided agents are sufficiently farsighted.

However, we have shown in the present paper that, once there are more than one seat to be allocated, more cooperation among students is needed to sustain the TTC matching as farsightedly stable.[18]

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A Appendix

A.1 Equitable Top Trading Cycles Algorithm

The Equitable Top Trading Cycles mechanism (Hakimov and Kesten, 2018) finds a matching by means of the following Equitable Top Trading Cycles algorithm (ETTC).

Step 1. Set $q_s^1 = q_s$ for all $s \in S$ where q_s^1 is the initial capacity of school s at Step 1.

1.A. (Inheritance) All seats are available to inherit in the first step and students are assigned seats according to the priority orders F to form student-school pairs. Let $F^{+1}(s,i) = \{j \in I \mid F_s(j) < F_s(i)\}$ be the set of students who have higher priority than student i for school s in Step 1. Let $\mathcal{IS}_1 = \{(i,s) \in I \times S \mid \#F^{+1}(s,i) \leq q_s^1\}$ be the set of student-school pairs formed by assigning

[18] In fact, one can show that Theorem 1 holds when restricting the deviations to coalitions involving at most two students. However, longer farsighted improving paths for reaching the TTC matching from any other matching would be required.

students one-by-one to the schools while respecting their capacities. In other words, $\mathcal{IS}_1$ consists of student-school pairs such that each school s pairs with q_s highest priority students.

1.B. (Pointing) Each student-school pair $(i,s) \in \mathcal{IS}_1$ points to the student-school pair $(i',s') \in \mathcal{IS}_1$ such that (1) s' is the best choice of student i in P_i, and (2) student i' has the highest priority in F_s among students who are assigned a seat at s', i.e. $(i,s) \mapsto (i',s')$ such that $F_s(i') < F_s(l)$ for any other $(l,s') \in \mathcal{IS}_1$. Notice that if $(i,s) \in \mathcal{IS}_1$ and s is the best choice school for i, then all pairs $(i,s') \in \mathcal{IS}_1$ point to (i,s). Since there is a finite number of students and schools, there is at least one cycle. Let $C_1 = \{c_1^1, c_1^2, \ldots, c_1^{L_1}\}$ be the set of cycles in Step 1.B where $L_1 \geq 1$ is the number of cycles in Step 1.B.

1.C. (Trading) If student i appears in the same cycle or in different cycles with different schools, then she is assigned a seat at her top choice among those schools. That is, for each $i \in I$ such that there exists $(i,s) \mapsto (i',s')$ in c_1^l and $(i,\hat{s}) \mapsto (i'',s'')$ in $c_1^{l'}$, possibly $c_1^l = c_1^{l'}$, $m_1(i) = s'$ such that $s' P_i s''$. The seats at all other schools than her top choice she points to in those cycles remain to be inherited. If student i only appears once in a cycle, say c_1^l, then she is matched with the school that is in the student-school pair she points to, i.e. if $(i,s) \mapsto (i',s')$ (possibly $s = s'$) in c_1^l, then $m_1(i) = s'$. Finally, if there is a student-school pair participating in a cycle, $(i,s) \in c_1^l$, and another student-school pair with the same student and a different school not participating at any cycle, $(i,s') \notin c_1^{l'}$, $c_1^{l'} \in C_1$, then this seat at school s' remains to be inherited.

Let $I_1 = \{i \in I \mid (i,s) \in c_1^l,\ c_1^l \in C_1\}$ be the set of students involved in a cycle in Step 1. Let $M_1 = \bigcup_{i \in I_1}(i, m_1(i))$ be all the matches formed between students and schools in Step 1. All student-school pairs that involve students who are matched in Step 1 are removed. Let $\mathcal{IS}_1^- = \{(i,s) \in \mathcal{IS}_1 \mid i \in I_1\}$ be the set of pairs from $\mathcal{IS}_1$ that are removed in Step 1. Let $\mathcal{IS}_1^+ = \{(i,s) \in \mathcal{IS}_1 \mid i \notin I_1\}$ be the set of pairs from $\mathcal{IS}_1$ that are not removed in Step 1. Let $S_1^+ = \{s \in S \mid (i,s) \in \mathcal{IS}_1^+,\ i \in I\}$ be the set of schools that were assigned in Step 1.A some student who are not matched in Step 1.C.

Let $\widehat{I}_1 = I \setminus I_1$ be the set of students who have not been assigned a seat at the end of Step 1. If $\widehat{I}_1 \neq \emptyset$, then go to Step 2.A. Otherwise, go to End.

Step $k \geq 2$. At the beginning of Step k, the remaining capacity of school s is q_s^k and the set of remaining students is $\widehat{I}_{k-1}$.

k.A. (Inheritance) Let $F^{+k}(s,i) = \{j \in \widehat{I}_{k-1} \mid F_s(j) < F_s(i)\}$ be the set of students who have higher priority than student i for school s in Step k. Let $\mathcal{IS}_k = \mathcal{IS}_{k-1}^+ \cup \{(i,s) \in \widehat{I}_{k-1} \times S \mid \#F^{+k}(s,i) \leq q_s^k \text{ and } s \notin S_{k-1}^+\}$ be the set of student-school pairs assigned in Step k.A where $\mathcal{IS}_{k-1}^+ = \{(i,s) \in \mathcal{IS}_{k-1} \mid i \notin I_{k-1}\}$ are the student-school pairs that were not removed in Step $k-1$ and $\{(i,s) \in \widehat{I}_{k-1} \times S \mid \#F^{+k}(s,i) \leq q_s^k \text{ and } s \notin S_{k-1}^+\}$ are the student-school pairs that are inherited and formed by assigning remaining students one-by-one to the schools while respecting their capacities.

k.B. (Pointing) Each student-school pair $(i,s) \in \mathcal{IS}_k$ points to the student-school pair $(i',s') \in \mathcal{IS}_k$ such that (1) s' is the best choice of student i in P_i, and (2) student i' has the highest priority in F_s among students that are assigned a seat at s', i.e. $(i,s) \mapsto (i',s')$ such that $F_s(i') < F_s(l)$ for any other $(l,s') \in \mathcal{IS}_k$. Since there is a finite number of students and schools, there is at least one cycle. Let $C_k = \{c_k^1, c_k^2, \ldots, c_k^{L_k}\}$ be the set of cycles in Step k.B where $L_k \geq 1$ is the number of cycles in Step k.B.

k.C. (Trading) If student i appears in the same cycle or in different cycles with different schools, then she is assigned a seat at her top choice among those schools. That is, for each $i \in \widehat{I}_{k-1}$ such that there exists $(i,s) \mapsto (i',s')$ in c_k^l and $(i,\hat{s}) \mapsto (i'',s'')$ in $c_k^{l'}$, possibly $c_k^l = c_k^{l'}$, $m_k(i) = s'$ such that $s' P_i s''$. The seats at all other schools than her top choice she points to in those cycles remain to be inherited. If student i only appears once in a cycle, say c_k^l, then she is matched with the school that is in the student-school pair she points to, i.e. if $(i,s) \mapsto (i',s')$ (possibly $s = s'$) in c_k^l, then $m_k(i) = s'$. Finally, if there is a student-school pair participating in a cycle, $(i,s) \in c_k^l$, and another student-school pair with the same student and a different school not participating at any cycle, $(i,s') \notin c_k^{l'}$, $c_k^{l'} \in C_k$, then this seat at school s' remains to be inherited.

Let $I_k = \{i \in \widehat{I}_{k-1} \mid (i,s) \in c_k^l,\, c_k^l \in C_k\}$ be the set of students involved in a cycle in Step k. Let $M_k = \bigcup_{i \in I_k} (i, m_k(i))$ be all the matches formed between students and schools in Step k. All student-school pairs that involve students who are matched in Step k are removed. Let $\mathcal{IS}_k^- = \{(i,s) \in \mathcal{IS}_k \mid i \in I_k\}$

be the set of pairs from $\mathcal{IS}_k$ that are removed in Step k. Let $\mathcal{IS}_k^+ = \{(i,s) \in \mathcal{IS}_k \mid i \notin I_k\}$ be the set of pairs from $\mathcal{IS}_k$ that are not removed in Step k. Let $S_k^+ = \{s \in S \mid (i,s) \in \mathcal{IS}_k^+,\, i \in I\}$ be the set of schools that were assigned in Step k.A some student who are not matched in Step k.C.

Let $\widehat{I}_k = \widehat{I}_{k-1} \setminus I_k$ be the set of students who have not been assigned a seat at the end of Step k. If $\widehat{I}_k \neq \emptyset$, then go to Step k+1.A. Otherwise, go to End.

End The algorithm stops when all students have been removed. Let $\bar{k} \geq 1$ be the step at which the algorithm stops. Let μ^E denote the matching obtained from the ETTC algorithm and it is given by $\mu^E = \bigcup_{k=1}^{\bar{k}} M_k$.

Proof of Theorem 2

Since $\{\mu^E\}$ is a singleton set, internal stability (IS) is satisfied. (ES) Take any matching $\mu \neq \mu^E$, we need to show that $\phi(\mu) \ni \mu^E$. We build in steps a farsighted improving path from μ to μ^E. Let $\widetilde{\mu}_0 = \mu$.

Step k.1. ($k \geq 1$) If $(i, \mu^E(i)) \in \widetilde{\mu}_{k-1}$ for all $i \in \{j \in I \mid (j,s) \in c_k^1\}$ and $1 \neq L_k$ then go to Step k.2 with $\mu'''_{k,1} = \widetilde{\mu}_{k-1}$. If $(i, \mu^E(i)) \in \widetilde{\mu}_{k-1}$ for all $i \in \{j \in I \mid (j,s) \in c_k^1\}$ and $1 = L_k$ then go to Step k.End with $\mu'''_{k,L_k} = \widetilde{\mu}_{k-1}$.

Otherwise, for each $i \in \{j \in I \mid (j,s) \in c_k^1\}$, let $c_k^1(i) = \{(i, s^{il'})\}_{l'=1}^{\tau(i,c_k^1)}$ be such that $(i, s^{il'}) \in c_k^1$ and $s^{il'} = s_{o_{l'}} \neq s^{il'+1} = s_{o_{l'+1}}$ with $o_{l'} < o_{l'+1}$ for $l' = 1, ..., \tau(i, c_k^1) - 1$. That is, $c_k^1(i)$ is an ordered set of the pairs involving student i in cycle c_k^1 where $\tau(i, c_k^1) = \#\{(j,s) \in c_k^1 \mid j = i\}$ is the number of distinct pairs involving student i in cycle c_k^1. Let $\tau(c_k^1) = \max_{i \in \{j \in I \mid (j,s) \in c_k^1\}} \tau(i, c_k^1)$.

Let $\Lambda_{k,1}(s) = \#\{(i,s') \notin \widetilde{\mu}_{k-1} \mid (i,s') = (i, s^{i1}) \text{ with } (i, s^{i1}) \in c_k^1(i) \text{ and } s' = s\}$ be the number of students who are not yet matched in $\widetilde{\mu}_{k-1}$ to school s that ranks them among the first q_s positions and is ranked first in their ordered set.

If $(i, \mu^E(i)) \notin \widetilde{\mu}_{k-1}$ for some $i \in \{j \in I \mid (j,s) \in c_k^1\}$ then $\mu'_{k,1} = \widetilde{\mu}_{k-1} - \{(i, \widetilde{\mu}_{k-1}) \mid (i,s) \in c_k^1 \text{ and } \widetilde{\mu}_{k-1}(i) \neq i\} + \{(i, s^{i1}) \mid i \in \{j \in I \mid (j,s) \in c_k^1\}\} - \{(j,s) \in \widetilde{\mu}_{k-1} \mid \Lambda_j^s(\widetilde{\mu}_{k-1}) < \Lambda_{k,1}(s) - q_s + \#\widetilde{\mu}_{k-1}(s)\}$ where $\Lambda_j^s(\widetilde{\mu}_{k-1}) = \#\{l \in I \mid (l,s) \in \widetilde{\mu}_{k-1} \text{ and } F_s(l) > F_s(j)\}$ is the number of students who are matched to school s in $\widetilde{\mu}_{k-1}$ and have a lower priority than student j. Looking forward towards μ^E students belonging to $\{j \in I \mid (j,s) \in c_k^1\}$ weakly prefer μ^E to $\widetilde{\mu}_{k-1}$ with at least one of them strictly preferring μ^E.

Next, if $(i, \mu'_{k,1}(i)) = (i, \mu^E(i))$ and $c_k^1 = \{(i, \mu^E(i))\}$ then $\mu''_{k,1} = \mu'_{k,1}$. Otherwise, $\mu''_{k,1} = \mu'_{k,1} - \{(i, s^{i1}) \mid i \in \{j \in I \mid (j,s) \in c_k^1\}\}$ so that all students involved in c_k^1 are unmatched. If $\tau(i, c_k^1) = 1$ for all $i \in \{j \in I \mid (j,s) \in c_k^1\}$ and $1 \neq L_k$, then go to Step k.2. If $\tau(i, c_k^1) = 1$ for all $i \in \{j \in I \mid (j,s) \in c_k^1\}$ and $1 = L_k$, then go to Step k.End with $\mu'''_{k,L_k} = \mu''_{k,1}$. If $\tau(i, c_k^1) \neq 1$ for some $i \in \{j \in I \mid (j,s) \in c_k^1\}$ then go to Step k.1.A.

Step k.1.A Take all $i \in \widetilde{I}^1(c_k^1) = \{j \in I \mid (j,s) \in c_k^1 \text{ and } \tau(j, c_k^1) > 1\}$, where $\widetilde{I}^1(c_k^1)$ is the set of all students who are involved more than once in cycle c_k^1. Let $P(s^{i2}) = \{(j,s) \mid j \in \widetilde{I}^1(c_k^1) \text{ and } s = s^{i2}\}$ be the set of student-school pairs involving school s^{i2} in cycle c_1^k. Let $\widehat{l}(s^{i2}) = \#\mu''_{k,1}(s^{i2}) + \#P(s^{i2})$. From $\mu''_{k,1}$, looking forward towards μ^E, each student $i \in \widetilde{I}^1(c_k^1)$ matches with school s^{i2}. In the case more students are assigned to some school than the number of available slots, then students with a lower priority are dropped off. Hence, from $\mu''_{k,1}$ we reach the matching

$$
\begin{aligned}
\mu^2_{k,1} &= \mu''_{k,1} + \{(i, s^{i2}) \mid i \in \widetilde{I}^1(c_k^1)\} \\
&- \Big\{(j_l, s^{i^12}) \mid F_{s^{i^12}}(j_l) > F_{s^{i^12}}(j') \text{ for all } j' \in \mu''_{k,1}(s^{i^12}),\, j' \neq j_l\Big\}_{l=1}^{\widehat{l}(s^{i^12})} \text{ if } \widehat{l}(s^{i^12}) \geq q_{s^{i^12}} \\
&- \Big\{(j_l, s^{i^22}) \mid F_{s^{i^22}}(j_l) > F_{s^{i^22}}(j') \text{ for all } j' \in \mu''_{k,1}(s^{i^22}),\, j' \neq j_l\Big\}_{l=1}^{\widehat{l}(s^{i^12})} \text{ if } \widehat{l}(s^{i^22}) \geq q_{s^{i^22}} \\
&\vdots \\
&- \Big\{(j_l, s^{i^{\bar{s}}2}) \mid F_{s^{i^{\bar{s}}2}}(j_l) > F_{s^{i^{\bar{s}}2}}(j') \text{ for all } j' \in \mu''_{k,1}(s^{i^{\bar{s}}2}),\, j' \neq j_l\Big\}_{l=1}^{\widehat{l}(s^{i^{\bar{s}}2})} \text{ if } \widehat{l}(s^{i^{\bar{s}}2}) \geq q_{s^{i^{\bar{s}}2}},
\end{aligned}
$$

where $\{s^{i^12}, s^{i^22}, \ldots, s^{i^{\bar{s}}2}\} = \{s \in S \mid s = s^{i2} \text{ for } i \in \widetilde{I}^1(c_k^1)\}$ and $\bar{s} = \#\{s \in S \mid s = s^{i2} \text{ for } i \in \widetilde{I}^1(c_k^1)\}$.

Looking forward towards μ^E students belonging to $\widetilde{I}^1(c_k^1)$ prefer μ^E to $\mu^2_{k,1}$. Next, each student $i \in \widetilde{I}^1(c_k^1)$ leaves her school s^{i2} to become unmatched and guaranteeing a free slot at school s^{i2}. We reach $\mu^{2'}_{k,1} = \mu^2_{k,1} - \{(i, s^{i2}) \mid i \in \widetilde{I}^1(c_k^1)\}$.

Next, take all $i \in \widetilde{I}^2(c_k^1) = \{j \in I \mid (j,s) \in c_k^1 \text{ and } \tau(j, c_k^1) > 2\}$, where $\widetilde{I}^2(c_k^1)$ is the set of all students who are involved more than twice in cycle c_k^1. Let $P(s^{i3}) = \{(j,s) \mid j \in \widetilde{I}^2(c_k^1) \text{ and } s = s^{i3}\}$ be the set of student-school pairs involving school s^{i3} in cycle c_k^1. Let $\widehat{l}(s^{i3}) = \#\mu^{2'}_{k,1}(s^{i3}) + \#P(s^{i3})$. From $\mu^{2'}_{k,1}$, looking forward towards μ^E, each student $i \in \widetilde{I}^2(c_k^1)$ matches with school s^{i3}.

In the case more students are assigned to some school than the number of available slots, then students with a lower priority are dropped off. Hence, from $\mu^{2'}_{k,1}$ we reach the matching

$$\begin{aligned}
&\mu^3_{k,1} = \mu^{2'}_{k,1} + \{(i, s^{i3}) \mid i \in \widetilde{I}^2(c^1_k)\} \\
&- \left\{(j_l, s^{i^1 3}) \mid F_{s^{i^1 3}}(j_l) > F_{s^{i^1 3}}(j') \text{ for all } j' \in \mu^{2'}_{k,1}(s^{i^1 3}),\, j' \neq j_l\right\}_{l=1}^{\widehat{l}(s^{i^1 3})} \text{ if } \widehat{l}(s^{i^1 3}) \geq q_{s^{i^1 3}} \\
&- \left\{(j_l, s^{i^2 3}) \mid F_{s^{i^2 3}}(j_l) > F_{s^{i^2 3}}(j') \text{ for all } j' \in \mu^{2'}_{k,1}(s^{i^2 3}),\, j' \neq j_l\right\}_{l=1}^{\widehat{l}(s^{i^1 3})} \text{ if } \widehat{l}(s^{i^2 3}) \geq q_{s^{i^2 3}} \\
&\vdots \\
&- \left\{(j_l, s^{i^{\bar{s}} 3}) \mid F_{s^{i^{\bar{s}} 3}}(j_l) > F_{s^{i^{\bar{s}} 3}}(j') \text{ for all } j' \in \mu^{2'}_{k,1}(s^{i^{\bar{s}} 3}),\, j' \neq j_l\right\}_{l=1}^{\widehat{l}(s^{i^{\bar{s}} 3})} \text{ if } \widehat{l}(s^{i^{\bar{s}} 3}) \geq q_{s^{i^{\bar{s}} 3}},
\end{aligned}$$

where $\{s^{i^1 3}, s^{i^2 3}, \ldots, s^{i^{\bar{s}} 3}\} = \{s \in S \mid s = s^{i3} \text{ for } i \in \widetilde{I}^2(c^1_k)\}$ and $\bar{s} = \#\{s \in S \mid s = s^{i3} \text{ for } i \in \widetilde{I}^2(c^1_k)\}$.

Looking forward towards μ^E students belonging to $\widetilde{I}^2(c^1_k)$ prefer μ^E to $\mu^3_{k,1}$. Next, each student $i \in \widetilde{I}^2(c^1_k)$ leaves her school s^{i3} to become unmatched and guaranteeing a free slot at school s^{i3}. We reach $\mu^{3'}_{k,1} = \mu^3_{k,1} - \{(i, s^{i3}) \mid i \in \widetilde{I}^2(c^1_k)\}$.

We repeat this process until we reach in the end the matching $\mu'''_{k,1} = \mu^{\tau(c^1_k)}_{k,1} - \{(i, s^{i\tau(c^1_k)}) \mid i \in \widetilde{I}^{\tau(c^1_k)-1}(c^1_k)\}$ where all students involved in c^1_k are unmatched and each school s involved in c^1_k has at least $\#\{(i, s') \in c^1_k \mid s' = s\}$ free slots.

Step $k.l.$ ($l > 1$) If $(i, \mu^E(i)) \in \mu'''_{k,l-1}$ for all $i \in \{j \in I \mid (j, s) \in c^l_k\}$ and $l \neq L_k$ then go to Step $k.l+1$ with $\mu'''_{k,l} = \mu'''_{k,l-1}$. If $(i, \mu^E(i)) \in \mu'''_{k,l-1}$ for all $i \in \{j \in I \mid (j, s) \in c^l_k\}$ and $l = L_k$ then go to Step k.End with $\mu'''_{k,L_k} = \mu'''_{k,l-1}$.

Otherwise, for each $i \in \{j \in I \mid (j, s) \in c^l_k\}$, let $c^l_k(i) = \{(i, s^{il'})\}_{l'=1}^{\tau(i, c^l_k)}$ be such that $(i, s^{il'}) \in c^l_k$ and $s^{il'} = s_{o_{l'}} \neq s^{il'+1} = s_{o_{l'+1}}$ with $o_{l'} < o_{l'+1}$ for $l' = 1, ..., \tau(i, c^l_k) - 1$. That is, $c^l_k(i)$ is an ordered set of the pairs involving student i in cycle c^l_k where $\tau(i, c^l_k) = \#\{(j, s) \in c^l_k \mid j = i\}$ is the number of distinct pairs involving student i in cycle c^l_k. Let $\tau(c^l_k) = \max_{i \in \{j \in I \mid (j,s) \in c^l_k\}} \tau(i, c^l_k)$.

Let $\Lambda_{k,l}(s) = \#\{(i, s') \notin \mu'''_{k,l-1} \mid (i, s') = (i, s^{i1}) \text{ with } (i, s^{i1}) \in c^l_k(i) \text{ and } s' = s\}$ be the number of students who are not yet matched in $\mu'''_{k,l-1}$ to school s that ranks them among the first q_s positions and is ranked first in their ordered set.

If $(i, \mu^E(i)) \notin \mu'''_{k,l-1}$ for some $i \in \{j \in I \mid (j,s) \in c^l_k\}$ then $\mu'_{k,l} = \mu'''_{k,l-1} - \{(i, \mu'''_{k,l-1}(i)) \mid (i,s) \in c^l_k \text{ and } \mu'''_{k,l-1}(i) \neq i\} + \{(i, s^{i1}) \mid i \in \{j \in I \mid (j,s) \in c^l_k\}\} - \{(j,s) \in \mu'''_{k,l-1} \mid \Lambda^s_j(\mu'''_{k,l-1}) < \Lambda_{k,l}(s) - q_s + \#\mu'''_{k,l-1}(s)\}$ where $\Lambda^s_j(\mu'''_{k,l-1}) = \#\{l' \in I \mid (l', s) \in \mu'''_{k,l-1} \text{ and } F_s(l') > F_s(j)\}$ is the number of students who are matched to school s in $\mu'''_{k,l-1}$ and have a lower priority than student j. Looking forward towards μ^E students belonging to $\{j \in I \mid (j,s) \in c^l_k\}$ weakly prefer μ^E to $\mu'''_{k,l-1}$ with at least one of them strictly preferring μ^E.

Next, if $(i, \mu'_{k,l}(i)) = (i, \mu^E(i))$ and $c^l_k = \{(i, \mu^E(i))\}$ then $\mu''_{k,l} = \mu'_{k,l}$. Otherwise, $\mu''_{k,l} = \mu'_{k,l} - \{(i, s^{i1}) \mid i \in \{j \in I \mid (j,s) \in c^l_k\}\}$ so that all students involved in c^l_k are unmatched. If $\tau(i, c^l_k) = 1$ for all $i \in \{j \in I \mid (j,s) \in c^l_k\}$ and $l \neq L_k$, then go to Step $k.l+1$. If $\tau(i, c^l_k) = 1$ for all $i \in \{j \in I \mid (j,s) \in c^l_k\}$ and $l = L_k$, then go to Step k.End with $\mu'''_{k,L_k} = \mu''_{k,l}$. If $\tau(i, c^l_k) \neq 1$ for some $i \in \{j \in I \mid (j,s) \in c^l_k\}$ then go to Step $k.l$.A.

Step $k.l$.A Take all $i \in \widetilde{I}^1(c^l_k) = \{j \in I \mid (j,s) \in c^l_k \text{ and } \tau(j, c^l_k) > 1\}$, where $\widetilde{I}^1(c^l_k)$ is the set of all students who are involved more than once in cycle c^l_k. Let $P(s^{i2}) = \{(j,s) \mid j \in \widetilde{I}^1(c^l_k) \text{ and } s = s^{i2}\}$ be the set of student-school pairs involving school s^{i2} in cycle c^l_k. Let $\widehat{l}(s^{i2}) = \#\mu''_{k,l}(s^{i2}) + \#P(s^{i2})$. From $\mu''_{k,l}$, looking forward towards μ^E, each student $i \in \widetilde{I}^1(c^l_k)$ matches with school s^{i2}. In the case more students are assigned to some school than the number of available slots, then students with a lower priority are dropped off. Hence, from $\mu''_{k,l}$ we reach the matching

$$\begin{aligned}
\mu^2_{k,l} &= \mu''_{k,l} + \{(i, s^{i2}) \mid i \in \widetilde{I}^1(c^l_k)\} \\
&- \left\{(j_{l'}, s^{i^12}) \mid F_{s^{i^12}}(j_{l'}) > F_{s^{i^12}}(j') \text{ for all } j' \in \mu''_{k,l}(s^{i^12}),\, j' \neq j_{l'}\right\}_{l'=1}^{\widehat{l}(s^{i^12})} \text{ if } \widehat{l}(s^{i^12}) \geq q_{s^{i^12}} \\
&- \left\{(j_{l'}, s^{i^22}) \mid F_{s^{i^22}}(j_{l'}) > F_{s^{i^22}}(j') \text{ for all } j' \in \mu''_{k,l}(s^{i^22}),\, j' \neq j_{l'}\right\}_{l'=1}^{\widehat{l}(s^{i^12})} \text{ if } \widehat{l}(s^{i^22}) \geq q_{s^{i^22}} \\
&\vdots \\
&- \left\{(j_{l'}, s^{i^{\bar{s}}2}) \mid F_{s^{i^{\bar{s}}2}}(j_{l'}) > F_{s^{i^{\bar{s}}2}}(j') \text{ for all } j' \in \mu''_{k,l}(s^{i^{\bar{s}}2}),\, j' \neq j_{l'}\right\}_{l'=1}^{\widehat{l}(s^{i^{\bar{s}}2})} \text{ if } \widehat{l}(s^{i^{\bar{s}}2}) \geq q_{s^{i^{\bar{s}}2}},
\end{aligned}$$

where $\{s^{i^12}, s^{i^22}, \ldots, s^{i^{\bar{s}}2}\} = \{s \in S \mid s = s^{i2} \text{ for } i \in \widetilde{I}^1(c^l_k)\}$ and $\bar{s} = \#\{s \in S \mid s = s^{i2} \text{ for } i \in \widetilde{I}^1(c^l_k)\}$.

Looking forward towards μ^E students belonging to $\widetilde{I}^1(c^l_k)$ prefer μ^E to $\mu^2_{k,l}$. Next, each student $i \in \widetilde{I}^1(c^l_k)$ leaves her school s^{i2} to become unmatched and guaranteeing a free slot at school s^{i2}. We reach $\mu^{2'}_{k,l} = \mu^{2}_{k,l} - \{(i, s^{i2}) \mid i \in \widetilde{I}^1(c^l_k)\}$.

Next, we repeat this process with students who are involved more than twice in cycle c^l_k until we reach in the end the matching $\mu'''_{k,l} = \mu^{\tau(c^l_k)}_{k,l} - \{(i, s^{i\tau(c^l_k)}) \mid i \in \widetilde{I}^{\tau(c^l_k)-1}(c^l_k)\}$ where all students involved in c^l_k are unmatched and each school s involved in c^l_k has at least $\#\{(i, s') \in c^l_k \mid s' = s\}$ free slots.

If $l \neq L_k$, then go to Step $k.l$+1. Otherwise, go to Step k.End with $\mu'''_{k,L_1} = \mu'''_{k,l}$.

Step k.End. We have reached μ'''_{k,L_k} where each student i involved in C_k is either matched to $\mu^E(i)$ or unmatched and each school s involved in C_k has at least $\#\{(i, s') \in \bigcup_{l=1}^{L_k} c^l_k \mid s' = s \text{ and } \mu'''_{k,L_k}(i) \neq \mu^E(i)\}$ free slots. Next, those unmatched students join the school they point to in C_k to form the matching $\widetilde{\mu}_k = \mu'''_{k,L_k} + \{(i, s) \in M_k \mid (i, s) \notin \mu'''_{k,L_k}\}$ so that each student i involved in C_k is matched to her school $\mu^E(i)$. If $\widetilde{\mu}_k = \mu^E$ then the process ends. Otherwise, go to Step k+1.

End. The process goes on until we reach $\widetilde{\mu}_{\bar{k}} = \bigcup_{k=1}^{\bar{k}} M_k = \mu^E$, where $\bar{k} \geq 1$.

□

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