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Limited Farsightedness in Priority-Based Matching

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ABSTRACT

We introduce the horizon-k vNM stable set to study one-to-one priority-based matching problems with limited farsightedness. We show that, once agents are sufficiently farsighted, the matching obtained from the Top Trading Cycles (TTC) algorithm becomes stable: a singleton set consisting of the TTC matching is a horizon-k vNM stable set if the degree of farsightedness is greater than three times the number of agents in the largest cycle of the TTC. Our main results do not hold per se for many-to-one priority-based matching problems: more coordination and cooperation on behalf of the agents are required. In the presence of couples, farsightedness may improve both efficiency and stability. When each agent owns an object, a singleton set consisting of the TTC matching is the unique horizon-k vNM stable set.

1 | Introduction

Many objects such as houses, school seats, jobs, or organs are allocated based on the preferences of the agents and their priorities.[1] Two prominent mechanisms used for priority-based matching are the Gale and Shapley's (1962) Deferred Acceptance (DA) mechanism and the Shapley and Scarf's (1974) Top Trading Cycles (TTC) mechanism, applied across the globe in different forms, such as public school choice, subsidized housing assignment, and cadaver organ allocation.[2] However, none of them satisfies two important properties: the TTC mechanism is Pareto-efficient while the DA mechanism may select an inefficient matching; the DA mechanism is stable while the TTC mechanism may select an unstable matching.[3]

Two approaches for analyzing the stability of matchings have been proposed in the literature depending on whether and how far agents anticipate that their actions may also induce others to change their matches. On the one hand, standard stability concepts involve fully myopic agents in the sense that they do not anticipate that others might react to their actions. On the other hand, a number of solution concepts involve perfectly farsighted agents who fully anticipate the complete sequence of reactions that results from their own actions.

However, experimental evidence suggests that subjects are consistent with an intermediate degree of farsightedness: agents only anticipate a limited number of reactions by the other agents to the actions that they take themselves. Moreover, recent experiments on network formation provide evidence in favor of a mixed population consisting of both myopic and (limited) farsighted individuals.[4] Farsighted behavior on one side of the market is observed in several priority-based matching problems. Public school teachers in France can apply every year to be transferred to another school. The transfer is done through a centralized mechanism, where teachers report a list of preferences over schools and priority rules determine who gets what. Priorities are based on a score with criteria set by law that vary over time depending on seniority but also, for instance, if a teacher has taught 5 years in a disadvantaged school. In this case, when a teacher must decide to apply to transfer from one school to another during her career, she takes into account how such decision and her former experience will impact her future score, and thus her chances for later transfers (see Combe et al. 2022). A patient in need of a kidney faces several options for treatment. One can wait to receive an offer of a deceased donor transplant, or one can rely on a compatible or incompatible living donor and join a Kidney Exchange Program to exchange her donor for a more compatible donor from another incompatible pair. The procedure for allocating deceased donor kidneys prioritizes certain types of patients, such as young or hyper-sensitized ones. The availability of a compatible or incompatible living donor impacts the decision of a patient on whether to accept a deceased donor kidney offer and her score in the priority list (see Ashlagi and Roth 2021).

Existing literature that examines the trade-off between stability and efficiency tends to adopt a myopic perspective. In one-to-one priority-based matching problems, is it possible to stabilize the matching obtained from the DA or TTC algorithm when agents are limited farsighted? If so, to what extent do agents need to be farsighted? What happens if there could be multiple units/copies of each object or if agents either form couples or own the objects?

To answer these questions, we introduce the notion of the horizon-k vNM stable set to study the matchings that are stable when agents are limited in their degree of farsightedness.[5] A horizon-k improving path for priority-based matching problems is a sequence of matchings that can emerge when limited farsighted agents form or destroy matches based on the improvement that the k-step ahead matching offers them relative to the current matching. Along a horizon-k improving path, one agent moves at a time and an agent can match to an object if either the object is unassigned or she has higher priority than the agent who was assigned to the object so far.[6] A set of matchings is a horizon-k vNM stable set if (Internal Stability) for any two matchings belonging to the set, there is no horizon-k improving path from one matching to the other one, and (External Stability) there always exists a horizon-k improving path from every matching outside the set to some matching within the set.

We show that, once agents are sufficiently farsighted, the matching obtained from the TTC algorithm becomes stable in one-to-one priority-based matching problems. Precisely, a singleton set consisting of the TTC matching is a horizon-k vNM stable set if the degree of farsightedness is greater than three times the number of agents in the largest cycle of the TTC. We provide a constructive proof where we build step by step a horizon-k improving path from any matching leading to the TTC matching. Along the horizon-k improving path, agents move one at a time and agents belonging to cycles sequentially act in the order of the formation of cycles in the TTC algorithm. Looking forward k steps ahead, agents belonging to a cycle first match one by one to the object that ranks them first on their priority list. Second, they give up one by one their object, and by doing so, vacating the object. Third, they match one by one to the object that they are assigned to in the TTC matching. The number of steps in this improving path is at most equal to three times the number of agents in the largest cycle of the TTC. Hence, looking forward such a number of steps ahead allows the agents to anticipate ending up with their TTC matches, and by doing so, they have incentives for engaging a move toward the matches that they have in the TTC matching. Thus, the matching obtained from the TTC algorithm is not only Pareto-efficient and strategy-proof but it is also horizon-k vNM stable. On the contrary, the matching obtained from the DA algorithm may not belong to any horizon-k vNM stable set for k large enough.

Notice that a singleton set consisting of the TTC matching is even a horizon-k vNM stable set for lower degrees of farsightedness. In fact, it is a horizon-k vNM stable set for k greater than or equal to the number of agents in the largest cycle plus two. However, such lower degrees of farsightedness require more coordination and cooperation on behalf of the agents along some horizon-k improving paths toward the TTC matching.

Our main results are robust to an alternative concept for limited farsightedness, which is obtained by adapting Herings et al. (2019)'s definition of a horizon-L farsighted set of networks to priority-based matching problems. This concept mainly replaces the internal stability condition of the horizon-k vNM stable set by two alternative conditions: deterrence of external deviations and minimality.

We next show that, once agents become farsighted, one has to distinguish between priority-based matching problems with multiple units of each object or multiple copies of each object. An object is said to be a copy of another object if both objects have the same priority list over the agents and all agents are indifferent between both objects. Our main results basically hold with multiple copies. However, in the presence of multiple units of each object, more coordination and cooperation on behalf of the agents are required to sustain the TTC matching as a singleton horizon-k vNM stable set: one needs to allow a group of agents to move all together along the horizon-k improving paths. Indeed, Atay et al. (2025) show that, for school choice problems (i.e., many-to-one priority-based matching problems) with farsighted students and coalitional deviations, a singleton set consisting of the TTC matching is a vNM farsighted stable set.[7]

In matching markets with couples, members of a couple do not only care about their own assignment but also about their partner's assignment. When all couples are myopic, a Pareto-dominated matching may be the unique stable matching even if there is some agent who has the highest priority for every object. In addition, there may not even exist a stable matching.[8] Once couples become farsighted, we find that, if there is some agent who has the highest priority for every object, then each Pareto-efficient matching where the couple of this agent gets their most preferred matches is a singleton horizon-k vNM stable set for k large enough. Thus, farsightedness may improve both efficiency and stability.

Finally, we show that, in the case of private endowments where each agent owns an object, a singleton set consisting of the TTC matching is the unique horizon-k vNM stable set if the degree of farsightedness is greater than three times the number of agents in the largest cycle of the TTC.

The paper is organized as follows: In Section 2, we introduce priority-based matching problems and we provide a formal description of the TTC mechanism and its algorithm. In Section 3, we introduce the notion of a horizon-k vNM stable set and we provide our main result. We also look at the robustness with respect to an alternative concept of limited farsightedness and multiple units/copies of each object. In Section 4, we consider priority-based matching problems with couples. In Section 5, we look at the matching problem where each agent owns an object. In Section 6, we conclude.

2 | Priority-Based Matching Problems

A priority-based matching problem is a list $\langle I, S, P, F\rangle$ where (i) $I = \{i_1, ..., i_n\}$ is the set of agents, (ii) $S = \{s_1, ..., s_m\}$ is the set of objects, (iii) $P = (P_{i_1}, ..., P_{i_n})$ is the preference profile where P_i is the strict preference of agent i over the objects and her outside option, and (iv) $F = (F_{s_1}, ..., F_{s_m})$ is the strict priority structure of the objects over the agents. Let i be a generic student and s be a generic object. We write i for singletons $\{i\} \subseteq I$ and s for singletons $\{s\} \subseteq S$. The preference P_i of agent i is a linear order over $S \cup i$. Agent i prefers object s to object s' if sP_is'. Object s is acceptable to agent i if sP_ii (i.e., sP_ii means that i strictly prefers s to her outside option). We often write $P_i = s, s', s''$ meaning that agent i's most preferred object is s, her second best is s', her third best is s'', and any other object is unacceptable for her. Let R_i be the weak preference relation associated with the strict preference relation P_i. The priority F_s of object s is a linear order over I. That is, F_s assigns ranks to agents according to their priority for object s. The rank of agent i for object s is denoted by $F_s(i)$ and $F_s(i) < F_s(j)$ means that agent i has higher priority for object s than agent j. For $s \in S, i \in I$, let $F^+(s, i) = \{j \in I | F_s(j) < F_s(i)\}$ be the set of agents who have higher priority than agent i for object s.

A matching outcome μ of a priority-based matching problem is a set of ordered pairs $\{(i, j)\}_{i \in I, j \in S \cup \{i\}}$ such that for all $i \in I$ and all $s \in S$ the following hold: (i) for each agent $i \in I$, there is a unique $j \in S \cup \{i\}$ with $(i, j) \in \mu$, and (ii) for each object $s \in S$, it holds that $\#\{i \in I | (i, s) \in \mu\} \leq 1$. For $i \in I$, we write $j = \mu(i)$ if $(i, j) \in \mu$; for $s \in S$, we write $i = \mu(s)$ if $(i, s) \in \mu$. Condition (i) means that agent i is assigned to object s under μ if $\mu(i) = s$ and is unassigned under μ if $\mu(i) = i$. Condition (ii) requires that no object is assigned to more than one agent. The set of all matchings is denoted $\mathcal{M}$.[9] For instance, $\mu = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, i_4)\}$ is the matching where agent i_1 is assigned to object s_2, agent i_2 is assigned to object s_3, agent i_3 is assigned to object s_1, and agent i_4 is unassigned.

A matching μ' Pareto-dominates a matching μ if $\mu'(i)R_i\mu(i)$ for all $i \in I$ and $\mu'(j)P_j\mu(j)$ for some $j \in I$. A matching is Pareto-efficient if it is not Pareto-dominated by another matching. A matching μ is stable if: (individual rationality) for all $i \in I$ we have $\mu(i)R_ii$; (non-wastefulness) for all $i \in I$ and all $s \in S$, if $sP_i\mu(i)$ then $\#\{j \in I | \mu(j) = s\} = 1$; and (no justified envy) for all $i, j \in I$ with $\mu(j) = s$, if $\mu(j)P_i\mu(i)$ then $j \in F^+(s, i)$.

A mechanism systematically selects a matching for any given priority-based matching problem $\langle I, S, P, F\rangle$.[10] Abdulkadiroğlu and Sönmez (2003) introduce the Top Trading Cycles (TTCs) mechanism for selecting a matching for general priority-based matching problems. In the case of a priority-based matching problem, the TTC mechanism finds a matching by means of the following TTC algorithm.

Step 1. Each agent $i \in I$ points to the object that is ranked first in P_i. If there is no such object, then agent i points to herself and she forms a self-cycle. Each object $s \in S$ points to the agent that has the highest priority in F_s. As the number of agents and objects are finite, there is at least one cycle. A cycle is an ordered list of distinct objects and distinct agents $(s^1, i^1, s^2, ..., s^l, i^l)$, where s^1 points to i^1 (denoted $s^1 \mapsto i^1$), i^1 points to s^2 ($i^1 \mapsto s^2$), s^l points to i^l ($s^l \mapsto i^l$), and i^l points to s^1 ($i^l \mapsto s^1$). Each object (agent) can be part of at most one cycle. Every agent in a cycle is assigned to the object that she points to and she is removed. Similarly, every agent in a self-cycle is not assigned to any object and is removed. If an object s is part of a cycle, then s is removed. Let $C_1 = \left\{c_1^1, c_1^2, ..., c_1^{L_1}\right\}$ be the set of cycles in Step 1 (where $L_1 \geq 1$ is the number of cycles in Step 1). Let I_1 be the set of agents who are assigned to some object at Step 1. Let S_1 be the set of objects that are assigned to some agent at Step 1. Let m_1^l be all the matches from cycle c_1^l that are formed in Step 1 of the algorithm:

$$m_1^l = \begin{cases} \left\{(i, s) | i, s \in c_1^l \text{ and } i \mapsto s\right\} & \text{if } c_1^l \neq (j) \\ \{(j, j)\} & \text{if } c_1^l = (j) \end{cases} \tag{1}$$

where (j, j) simply means that agent j who is in a self-cycle ends up being definitely unassigned to any object. Let $M_1 = \bigcup_{l=1}^{L_1} m_1^l$ be all the matches between agents and objects formed in Step 1 of the algorithm. Let $\bar{c}_1^l = \#\left\{i \in I | i \in c_1^l\right\}$ be the number of agents involved in cycle c_1^l, for $l = 1, ..., L_1$. Let $c_1^{\max} = \max\left\{\bar{c}_1^1, ..., \bar{c}_1^{L_1}\right\}$.

Step $k \geq 2$. Each remaining agent $i \in I \setminus \bigcup_{l=1}^{k-1} I_l$ points to the object $s \in S \setminus \bigcup_{l=1}^{k-1} S_l$ that is ranked first in P_i. If there is no such object, then agent i points to herself and she forms a self-cycle. Each object $s \in S \setminus \bigcup_{l=1}^{k-1} S_l$ points to the agent $j \in I \setminus \bigcup_{l=1}^{k-1} I_l$ that has the highest priority in F_s. There is at least one cycle. Every agent in a cycle is assigned to the object that she points to and she is removed. Similarly, every agent in a self-cycle is not assigned to any object and is removed. If an object s is part of a cycle, then s is removed. Let $C_k = \left\{c_k^1, c_k^2, ..., c_k^{L_k}\right\}$ be the set of cycles in Step k (where $L_k \geq 1$ is the number of cycles in Step k). Let I_k be the set of agents who are assigned to some object at Step k. Let S_k be the set of objects that are assigned to some agent at Step k. Let m_k^l be all the matches from cycle c_k^l that are formed in Step k of the algorithm:

$$m_k^l = \begin{cases} \left\{(i, s) | i, s \in c_k^l \text{ and } i \mapsto s\right\} & \text{if } c_k^l \neq (j) \\ \{(j, j)\} & \text{if } c_k^l = (j) \end{cases} \tag{2}$$

Let $M_k = \bigcup_{l=1}^{L_k} m_k^l$ be all the matches between agents and objects formed in Step k of the algorithm. Let $\bar{c}_k^l = \#\left\{i \in I | i \in c_k^l\right\}$ be the number of agents involved in cycle c_k^l, for $l = 1, ..., L_k$. Let $c_k^{\max} = \max\left\{\bar{c}_k^1, ..., \bar{c}_k^{L_k}\right\}$.

End. The algorithm stops when all agents have been removed. Let $\bar{k}$ be the step at which the algorithm stops. Let μ^T denote the matching obtained from the Top Trading Cycles mechanism and it is given by $\mu^T = \bigcup_{k=1}^{\bar{k}} M_k$. Let $\gamma = \max\left\{c_1^{\max}, ..., c_{\bar{k}}^{\max}\right\}$ be the maximum number of agents involved in any cycle of the TTC.

One property of the TTC algorithm is that, for any $k' \in \{1, ..., \bar{k} - 1\}$, given all the matches already settled, i.e., $\bigcup_{k=1}^{k'} M_k$, agents involved in cycle $c_{k'+1}^l$, $l \in \{1, ..., L_{k'+1}\}$, of Step $k' + 1$ of the TTC algorithm obtain their best possible assignment in $m_{k'+1}^l$. Hence, the TTC mechanism is Pareto-efficient. In addition, this mechanism is also strategy-proof (see Abdulkadiroğlu and Sönmez 2003). Meanwhile, TTC is individually rational and non-wasteful; it is not stable.[11] Another mechanism that is commonly adopted all over the world is Gale and Shapley's Deferred Acceptance (DA) algorithm. Let μ^D denote the matching obtained from the DA mechanism. The DA mechanism is strategy-proof and stable but not Pareto-efficient.[12]

3 | Horizon-*k* Vnm Stable Set

3.1 | Definitions

Is it possible to stabilize the matching obtained from the TTC algorithm once agents become limited farsighted? If yes, how much farsightedness from the agents do we need? To answer this question, we propose the notion of a horizon-k vNM stable set for priority-based matching problems to study the matchings that are stable when agents are limited in their degree of farsightedness.

A horizon-k improving path for priority-based matching problems is a sequence of matchings that can emerge when limited farsighted agents form or destroy matches based on the improvement that the k-steps ahead matching offers them relative to the current matching. A set of matchings is a horizon-k vNM stable set if (Internal Stability) for any two matchings belonging to the set, there is no horizon-k improving path from one matching to the other one, and (External Stability) there always exists a horizon-k improving path from every matching outside the set to some matching within the set.

Given a matching $\mu \in \mathcal{M}$ with agent $i \in I$ assigned to object $s \in S$, so $\mu(i) = s$, the matching μ' that is identical to μ, except that the match between i and s has been destroyed by i, is denoted by $\mu' = \mu - (i, s) + (i, i)$. Given a matching $\mu \in \mathcal{M}$ such that $i \in I$ and $s \in S$ are not matched to one another, the matching μ' that is identical to μ, except that the pair (i, s) has formed at μ', is denoted by $\mu' = \mu + (i, s) - (i, \mu(i)) - \{(j, s) | \mu(j) = s\} + \{(j, j) | \mu(j) = s\}$.[13]

Definition 1. Let $\langle I, S, P, F\rangle$ be a priority-based matching problem. A horizon-k improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \backslash \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, ..., \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, ..., L - 1\}$ either

i. $\mu_{l+1} = \mu_l - (i, s) + (i, i)$ for some $(i, s) \in I \times S$ such that $\mu_{\min\{l+k, L\}}(i) P_i \mu_l(i)$, or

ii. $\mu_{l+1} = \mu_l + (i, s) - (i, \mu_l(i)) - \{(j, s) | \mu_l(j) = s\} + \{(j, j) | \mu_l(j) = s\}$ for some $(i, s) \in I \times S$ such that $\mu_{\min\{l+k, L\}}(i) P_i \mu_l(i)$ and $F_s(i) < F_s(j)$ if $\mu_l(s) = j$.

Definition 1 tells us that a horizon-k improving path for priority-based matching problems consists of a sequence of matchings where along the sequence, agents form or destroy matches based on the improvement that the k-steps ahead matching offers them relative to the current one. Precisely, along a horizon-k improving path, each time some agent i is on the move, she is comparing her current match (i.e., $\mu_l(i)$) with the match that she will get k steps ahead on the sequence (i.e., $\mu_{l+k}(i)$), except if the end matching of the sequence lies within her horizon (i.e., $L < l + k$). In such a case, she simply compares her current match (i.e., $\mu_l(i)$) with the end match (i.e., $\mu_L(i)$).

Objects can be assigned to any agent on their priority lists unless they have already been assigned to some agent. When an object $s \in S$ is already assigned to some agent $\mu_l(s)$ at μ_l, this object s can be reassigned to another agent $\mu_{l+1}(s) \neq \mu_l(s)$ at μ_{l+1} only if agent $\mu_{l+1}(s)$ has a higher priority than agent $\mu_l(s)$.

Let some $\mu \in \mathcal{M}$ be given. If there exists a horizon-k improving path from a matching μ to a matching μ', then we write $\mu \to_k \mu'$. The set of matchings $\mu' \in \mathcal{M}$ such that there is a horizon-k improving path from μ to μ' is denoted by $\phi_k(\mu)$, so $\phi_k(\mu) = \{\mu' \in \mathcal{M} | \mu \to_k \mu'\}$.

Definition 2. Let $\langle I, S, P, F\rangle$ be a priority-based matching problem. A set of matchings $V \subseteq \mathcal{M}$ is a horizon-k vNM stable set if it satisfies:

i. For every $\mu, \mu' \in V$, it holds that $\mu' \notin \phi_k(\mu)$.

ii. For every $\mu \in \mathcal{M} \backslash V$, it holds that $\phi_k(\mu) \cap V \neq \emptyset$.

Condition (i) of Definition 2 corresponds to internal stability (IS). For any two matchings μ and μ' in the horizon-k vNM stable set V, there is no horizon-k improving path connecting μ to μ'. Condition (ii) of Definition 2 expresses external stability (ES). There exists a horizon-k improving path from every matching μ outside the horizon-k vNM stable set V to some matching in V.[14]

3.2 | Main Result

Remember that $\gamma = \max\left\{c_1^{\max}, ..., c_{\bar{k}}^{\max}\right\}$ is the maximum number of agents involved in any cycle of the TTC.

Theorem 1. *Let $\langle I, S, P, F\rangle$ be a priority-based matching problem and μ^T be the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is a horizon-k vNM stable set for $k \geq (3\gamma - 1)$.*

Theorem 1 shows that, once agents are sufficiently farsighted (i.e., $k \geq 3\gamma - 1$), the matching obtained from the TTC algorithm is stabilized. All the proofs not in the main text can be found in the appendix. In Example 1, we provide the basic intuition behind Theorem 1 and its proof. In addition, it shows that, once agents are no more myopic, the matching obtained from the DA algorithm may become unstable.

Example 1. Consider a priority-based matching problem $\langle I, S, P, F \rangle$ with $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2, s_3\}$. Agents' preferences and objects' priorities are as follows.[15]

Agents		
P_{i_1}	P_{i_2}	P_{i_3}
s_1	s_1	s_2
s_3	s_2	s_1
s_2	s_3	s_3

Objects		
F_{s_1}	F_{s_2}	F_{s_3}
i_3	i_2	i_2
i_1	i_1	i_3
i_2	i_3	i_1

In Example 1, $\mu^T = \{(i_1, s_3), (i_2, s_1), (i_3, s_2)\}$ is the matching obtained from the TTC algorithm. In the first round of the TTC algorithm, there is one cycle where agent i_2 points to object s_1, object s_1 points to agent i_3, agent i_3 points to object s_2, and object s_2 points to agent i_2. That is, $C_1 = \{c_1^1\}$ with $c_1^1 = (s_1, i_3, s_2, i_2)$. Agent i_2 is assigned to object s_1 and agent i_3 is assigned to object s_2: $m_1^1 = \{(i_2, s_1), (i_3, s_2)\}$, and so i_2 and i_3 exchange their priority. In the second round of the TTC algorithm, there is only one leftover agent, i_1, who points to object s_3, and one leftover object, s_3, that points to agent i_1. That is, $C_2 = \{c_2^1\}$ with $c_2^1 = (s_3, i_1)$. Agent i_1 is assigned to object s_3: $m_2^1 = \{(i_1, s_3)\}$, and so $\mu^T = m_1^1 \cup m_2^1$.

From Theorem 1, we know that $\{\mu^T\}$ is a horizon-k vNM stable set for $k \geq 3\gamma - 1$. Indeed, if $k \geq 3\gamma - 1$, then from any $\mu \neq \mu^T$, there exists a horizon-k improving path leading to μ^T. Notice that $\gamma = c_1^{\max} = 2$. Take, for instance, the matching $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$. We now construct a horizon-k improving path from μ to $\mu^T = \{(i_1, s_3), (i_2, s_1), (i_3, s_2)\}$ following the steps as in the proof of Theorem 1 that can be found in the appendix. This horizon-k improving path consists of a sequence of distinct matchings, $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ with

$$\begin{aligned}
\mu_0 &= \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\} = \mu, \\
\mu_1 &= \{(i_1, i_1), (i_2, s_2), (i_3, s_1)\}, \\
\mu_2 &= \{(i_1, i_1), (i_2, s_2), (i_3, i_3)\} \\
\mu_3 &= \{(i_1, i_1), (i_2, s_1), (i_3, i_3)\}, \\
\mu_4 &= \{(i_1, i_1), (i_2, s_1), (i_3, s_2)\}, \\
\mu_5 &= \{(i_1, s_3), (i_2, s_1), (i_3, s_2)\} = \mu^T.
\end{aligned}$$

First, we consider agents and objects belonging to the cycles in C_1. Notice that $m_1^1 = \{(i_2, s_1), (i_3, s_2)\} \cap \mu_0 = \varnothing$. Looking forward toward μ_4 and μ^T (where $\mu_4(i_3) = \mu^T(i_3)$), agent i_3 matches to the object s_1 that ranks her first on its priority list to reach the matching $\mu_1 = \{(i_1, i_1), (i_2, s_2), (i_3, s_1)\}$ where agent i_3 is matched to the object in c_1^1 where she has priority. By doing so, agent i_1 is left without object. In μ_0 (and μ_1), agent i_2 is already matched to the object in c_1^1 where she has priority.[16] Next, agent i_3 leaves her object s_1 to reach the matching $\mu_2 = \{(i_1, i_1), (i_2, s_2), (i_3, i_3)\}$ where agent i_3 is not assigned to any object. Agent i_3 is temporarily worse off, but she anticipates to end up being matched with $\mu^T(i_3)$. Next, agent i_2 matches to s_1 that was left unassigned to reach the matching $\mu_3 = \{(i_1, i_1), (i_2, s_1), (i_3, i_3)\}$. Next, agent i_3 matches to s_2 that was left by i_2 to reach the matching $\mu_4 = \{(i_1, i_1), (i_2, s_1), (i_3, s_2)\}$ with $m_1^1 = \{(i_2, s_1), (i_3, s_2)\} \subseteq \mu_4$. Finally, we consider agents and objects belonging to the cycles in C_2. As $m_2^1 = \{(i_1, s_3)\} \cap \mu_4 = \varnothing$, agent i_1 is assigned to object s_3 to form the match (i_1, s_3) and to reach the matching $\mu_5 = \mu^T$. Thus, $\mu^T \in \phi_k(\mu_0)$.

We now show that the matching $\mu^D = \{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$ obtained from the DA algorithm as well as the matching obtained from the Immediate Acceptance (IA) algorithm, $\mu^B = \{(i_1, s_1), (i_2, s_3), (i_3, s_2)\} \neq \mu^D$, do not belong to any horizon-k vNM stable set for $k \geq 5$. As agent i_1 is as well off and agents i_2 and i_3 are strictly better off in μ^T than in μ^D, we have that there is no horizon-k improving path from μ^T to μ^D for $k \geq 5$. That is, $\mu^D \notin \phi_k(\mu^T)$ for $k \geq 5$. Hence, $\{\mu^D\}$ is not a horizon-k vNM stable set for $k \geq 5$, as (ES) is violated. Let

$$\begin{aligned}
\mu^1 &= \{(i_1, s_1), (i_2, i_2), (i_3, s_2)\}, \\
\mu^2 &= \{(i_1, s_1), (i_2, s_3), (i_3, s_2)\} = \mu^B, \\
\mu^3 &= \{(i_1, s_1), (i_2, s_2), (i_3, i_3)\}, \\
\mu^4 &= \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}, \\
\mu^5 &= \{(i_1, i_1), (i_2, s_2), (i_3, s_1)\}.
\end{aligned}$$

Computing the horizon-k improving paths emanating from μ^T for $k \geq 5$, we obtain $\phi_k(\mu^T) = \{\mu^1, \mu^2, \mu^3, \mu^4\}$. Notice that $\mu^5 \notin \phi_k(\mu^T)$ as agent i_1 is worse off in μ^5 than in μ^T. From $\mu^1, \mu^2, \mu^3, \mu^4$ and μ^5, there is a horizon-k improving path to μ^D, i.e., $\mu^D \in \phi_k(\mu)$ for $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$. From μ^D, there is only a horizon-k improving path to μ^T, i.e., $\phi_k(\mu^D) = \{\mu^T\}$ for $k \geq 5$. For a set $V \supseteq \{\mu^D\}$ to be a horizon-k vNM stable set, we need that (i) $\mu^T \notin V$ (otherwise (IS) is violated), (ii) a single $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4\}$ should belong to V to satisfy (ES), as $\mu^D \notin \phi_k(\mu^T)$ for $k \geq 5$. However, V would then violate (IS) as $\mu^D \in \phi_k(\mu)$ for $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5\}$. Thus, there is no horizon-k vNM stable set V such that $\mu^D \in V$ for $k \geq 5$.

Is $V = \{\mu^T\}$ the unique horizon-k vNM stable set for $k \geq 5$ in our example? Any other set V' such that $\mu^T \in V'$ violates (IS), and hence, $\mu^T \notin V'$. Then, $\mu^D \in V'$ because, otherwise, V' violates (ES) as $\phi_k(\mu^D) = \{\mu^T\}$ for $k \geq 5$. As already shown, there is no V' such that $\mu^D \in V'$ that is a horizon-k vNM stable set for $k \geq 5$. Hence, we have that $V = \{\mu^T\}$ is the unique horizon-k stable set for $k \geq 5$.

Thus, the DA matching μ^D and the IA matching μ^B do not belong to any horizon-k vNM stable set for $k \geq 5$. As the matching obtained from the IA algorithm is Pareto-efficient, Example 1 also shows that there are priority-based matching problems where some Pareto-efficient matching does not belong to any horizon-k vNM stable set for $k \geq 3\gamma - 1$.

What happens if k becomes small? Computing the horizon-k improving paths emanating from μ^T for $k \leq 4$ in Example 1, we obtain $\phi_k(\mu^T) = \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^D\}$. So, there is now a horizon-k improving path from μ^T to μ^D. In addition, from $\mu^1, \mu^2, \mu^3, \mu^4$ and μ^5, there is still a horizon-k improving path to μ^D. That is, $\mu^D \in \phi_k(\mu)$ for $\mu \in \{\mu^1, \mu^2, \mu^3, \mu^4, \mu^5, \mu^T\}$ for $k \leq 4$. From μ^D, there is no horizon-k improving path for $k \leq 2$, but there is one for $3 \leq k \leq 4$, i.e., $\phi_k(\mu^D) = \varnothing$ for $k \leq 2$ and $\phi_k(\mu^D) = \{\mu^T\}$ for $3 \leq k \leq 4$. It follows then that for $3 \leq k \leq 4$, both $V = \{\mu^D\}$ and $V' = \{\mu^T\}$ are horizon-k vNM stable sets. However, for $k \leq 2$, $V = \{\mu^D\}$ is the unique horizon-k vNM stable set. In general, it holds that for any priority-based matching problem, the DA matching μ^D belongs to all horizon-1 vNM stable sets.[17]

Remark 1. Let $\langle I, S, P, F \rangle$ be a priority-based matching problem and μ^D be the matching obtained from the DA mechanism. The matching μ^D belongs to all horizon-1 vNM stable sets but may not belong to any horizon-k vNM stable set for $k \geq 3\gamma - 1$.

In the appendix, we show that one could find a tighter bound on k such that for all $k' \geq k$, the singleton set $\{\mu^T\}$ is a horizon-k' vNM stable set. By carefully choosing the order in which agents successively first match to their priority object, and next match to their top choice object, the singleton set $\{\mu^T\}$ is a horizon-k' vNM stable set for $k' \geq \gamma + 2$. However, it relies on improving paths that require much more coordination on behalf of the agents than the ones associated to the lower bound of Theorem 1, i.e., $3\gamma - 1$. Indeed, along those improving paths, the order in which agents finally match to their top choice object has to be the same as the order in which they first match to their priority object.

Remark 2. Let $\langle I, S, P, F \rangle$ be a priority-based matching problem and μ^T be the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is a horizon-k vNM stable set for $k \geq \gamma + 2$.

By simply replacing $\mu_{\min\{l+k,L\}}(i) P_i \mu_l(i)$ by $\mu_L(i) P_i \mu_l(i)$ in the definition of a horizon-k improving path, we obtain the definition of a farsighted improving path. Let $\phi_\infty(\mu)$ be the set of matchings that can be reached by means of a farsighted improving path emanating from μ. Given that the number of possible matchings is finite, there exists $k^\star$ such that for all $k \geq k^\star$, $\phi_k(\mu) = \phi_{k+1}(\mu)$, and so $\phi_{k^\star}(\mu) = \phi_\infty(\mu)$.[18]

Definition 3. Let $\langle I, S, P, F \rangle$ be a priority-based matching problem. A set of matchings $V \subseteq \mathcal{M}$ is a vNM farsighted stable set if it satisfies:

- **i.** For every $\mu, \mu' \in V$, it holds that $\mu' \notin \phi_\infty(\mu)$.
- **ii.** For every $\mu \in \mathcal{M} \backslash V$, it holds that $\phi_\infty(\mu) \cap V \neq \varnothing$.

Corollary 1. *Let $\langle I, S, P, F \rangle$ be a priority-based matching problem and μ^T be the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is a vNM farsighted stable set.*

Notice that our notion of vNM farsighted stable set reverts to the notion of myopic-farsighted stable set for one-to-one matching problems (introduced by Herings et al. 2020) when one side is farsighted and the other side is myopic. Doğan and Ehlers (2025) show the existence of myopic-farsighted stable sets in one-to-one matching problems. In addition, they find that the singleton set $\{\mu^E\}$ is a myopic-farsighted stable set, where μ^E is the matching obtained from the EADA mechanism (see Kesten 2010).[19]

The notion of vNM farsighted stable set does not require that agents choose their best alternative along the farsighted improving paths. Nevertheless, Corollary 1 is robust to the incorporation of various forms of maximality in the definition of farsighted improving path, like the strong rational expectations farsighted stable set in Dutta and Vohra (2017) and absolute maximality as in Ray and Vohra (2019).

3.3 | Multiple Copies Versus Multiple Units

We now highlight the importance of distinguishing between priority-based matching problems with multiple units of each object or multiple copies of each object. An object is said to be a copy of another object if both objects have the same priority list over the agents and all agents are indifferent between both objects. Theorem 1 and its remarks hold with multiple copies. However, as shown in the following example, in the presence of multiple units of each object, one needs to allow group of agents to move all together to sustain the TTC matching as a singleton horizon-k vNM stable set.

Example 2. Consider a priority-based matching problem $\langle I, S, R, F \rangle$ with $I = \{i_1, i_2, i_3, i_4\}$ and $S = \left\{s_1, s_1' s_2, s_3\right\}$. All agents are indifferent between s_1 and s_1'. Both objects s_1 and s_1' rank the agents in the same order. Let q_s be the number of units of each object s. Agents' preferences and objects' priorities are as follows.

Agents

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
s_1, s_1'	s_1, s_1'	s_2	s_1, s_1'
s_2	s_2	s_1, s_1'	s_2
s_3	s_3	s_3	s_3

Objects

	F_{s_1}	$F_{s_1'}$	F_{s_2}	F_{s_3}
q_s	1 (2)	1 (0)	1 (1)	1 (1)
	i_3	i_3	i_1	i_2
	i_1	i_1	i_2	i_3
	i_4	i_4	i_4	i_4
	i_2	i_2	i_3	i_1

Consider first the matching problem with multiple copies of s_1, where $q_{s_1} = 1$ and $q_{s_1'} = 1$.[20] Depending on whether agent i_1 points either to s_1 or to s_1', the TTC mechanism leads either to $\mu^T = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_1')\}$ or to $\mu^{T'} = \{(i_1, s_1'), (i_2, s_3), (i_3, s_2), (i_4, s_1)\}$. One can show that, from any $\mu \neq \mu^T, \mu^{T'}$, there is a horizon-k improving path going to either μ^T or $\mu^{T'}$. For instance, take $\mu = \left\{(i_1, s_1), (i_2, s_2), (i_3, s_3), \left(i_4, s_1'\right)\right\}$; the horizon-$k$ improving path consists of a sequence of distinct matchings, $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6$ with

$$
\begin{aligned}
\mu_0 &= \left\{(i_1, s_1), (i_2, s_2), (i_3, s_3), \left(i_4, s_1'\right)\right\}, \\
\mu_1 &= \left\{(i_1, i_1), (i_2, s_2), (i_3, s_1), \left(i_4, s_1'\right)\right\}, \\
\mu_2 &= \left\{(i_1, s_2), (i_2, i_2), (i_3, s_1), \left(i_4, s_1'\right)\right\}, \\
\mu_3 &= \left\{(i_1, i_1), (i_2, i_2), (i_3, s_1), \left(i_4, s_1'\right)\right\}, \\
\mu_4 &= \left\{(i_1, i_1), (i_2, i_2), (i_3, s_2), \left(i_4, s_1'\right)\right\}, \\
\mu_5 &= \left\{(i_1, s_1), (i_2, i_2), (i_3, s_2), \left(i_4, s_1'\right)\right\}, \\
\mu_6 &= \left\{(i_1, s_1), (i_2, s_3), (i_3, s_2), \left(i_4, s_1'\right)\right\} = \mu^T,
\end{aligned}
$$

It follows that the set $\{\mu^T, \mu^{T'}\}$ is a horizon-k vNM stable set for $k \geq 5$. In the case agents are indifferent between some objects and these objects rank the agents in the same order, the proof of Theorem 1 and Corollary 1 can be basically extended to show that the set of TTC matchings[21] is a horizon-k vNM stable set and a vNM farsighted stable set.

Consider next the matching problem with multiple units of s_1, where $q_{s_1} = 2$ and $q_{s_1'} = 0$. The TTC mechanism leads to $\mu^T = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_1)\}$ with cycles $C_1 = \{(s_1, i_3, s_2, i_1)\}$, $C_2 = \{(s_1, i_4)\}$ and $C_3 = \{(s_3, i_2)\}$. We now argue that the set $\{\mu^T\}$ is no more a horizon-k vNM stable set. From the matching $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_1)\}$, there is no horizon-$k$ improving path leading to μ^T. At μ, i_1 is already matched to her preferred object and i_3 cannot evict i_1 from s_1 so that i_1 would next evict i_2 from s_2. Indeed, i_4 is matched to s_1 and i_4 is ranked below i_1 by s_1. However, if agents i_1 and i_3, who look k steps ahead toward μ^T, were able to deviate together, then they would first match to their prioritized objects, they would next leave those objects, and they would finally match their TTC objects. Notice that, i_1 is indifferent between μ and μ^T, whereas i_3 strictly prefers μ^T to μ. In fact, Atay et al. (2025) show that, in general, $\{\mu^T\}$ is a horizon-k vNM stable set for priority-based matching problems with multiple units if we allow for such coalitional moves. However, coalitional moves require more coordination and cooperation on behalf of the farsighted agents. Without coalitional moves, the singleton set $\{\mu^E\}$ is the unique horizon-k vNM stable set for $k > 4$, where $\mu^E = \{(i_1, s_1), (i_2, s_1), (i_3, s_2), (i_4, s_3)\}$ is the matching obtained from the EADA mechanism. The set $\{\mu^E\}$ satisfies external stability (ES), as there are horizon-k improving paths from each of the following matchings:

$$
\begin{aligned}
\mu^D &= \{(i_1, s_1), (i_2, s_2), (i_3, s_1), (i_4, s_3)\}, \\
\mu^T &= \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_1)\}, \\
\mu^\star &= \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_1)\}, \\
\mu' &= \{(i_1, s_1), (i_2, s_1), (i_3, s_3), (i_4, s_2)\}, \\
\mu'' &= \{(i_1, s_1), (i_2, s_3), (i_3, s_1), (i_4, s_2)\},
\end{aligned}
$$

leading to μ^E. For instance, from μ^T agent i_2 first matches to s_2 and evicts agent i_3 from s_2. Next, agent i_3 matches to s_1 and evicts agent i_4 from s_1. Next, agent i_3 leaves s_1. Next, agent i_2 matches to s_1. Next, agent i_3 matches to s_2. Next, i_4 matches to s_3, and so we reach μ^E. From any matching where i_1 is not matched to s_1, agent i_1 first matches to s_1. Along those horizon-k improving paths toward μ^E, it is sufficient that all agents look five steps or more ahead for engaging a move.[22] No other set can be a horizon-k vNM stable set. First, any other singleton set violates ES. Second, any set containing matchings where i_1 is not matched to s_1 together with only one of the above matchings also violates ES. Thus, at least two matchings from $\{\mu^D, \mu^T, \mu^\star, \mu', \mu''\}$ should belong to a candidate horizon-k vNM stable set. Third, any set containing at least two matchings from $\{\mu^D, \mu^T, \mu^\star, \mu', \mu''\}$ violates either internal stability (IS) or ES. Any set with two matchings from $\{\mu^D, \mu^T, \mu^\star, \mu', \mu''\}$ that violates IS, would also violates IS if we add other matchings to the set. Any set containing at least two matchings from $\{\mu^D, \mu^T, \mu^\star, \mu', \mu''\}$ that violates ES would violate IS if we add the matching(s) that is(are) needed to satisfy ES. It then follows that $\{\mu^E\}$ is the unique horizon-k vNM stable set.

In Example 2, when all agents are myopic, stable matchings are either both matchings $\left\{(i_1, s_1), (i_2, s_2), \left(i_3, s_1'\right), (i_4, s_3)\right\}$ and $\left\{\left(i_1, s_1'\right), (i_2, s_2), (i_3, s_1), (i_4, s_3)\right\}$ or the single equivalent matching $\{(i_1, s_1), (i_2, s_2), (i_3, s_1), (i_4, s_3)\}$. However, when all agents become farsighted, it matters for stability whether the matching problem involves multiple copies or multiple units and whether coalitional moves are allowed or not.

3.4 | Alternative Notion of Limited Farsightedness

An alternative concept for limited farsightedness is obtained by adapting Herings et al. (2019)'s definition of a horizon-L farsighted set of networks to priority-based matching problems (see the appendix for details). A horizon-L farsighted set of matchings V has to satisfy three requirements: (i) horizon-L deterrence of external deviations, (ii) horizon-L external stability, and (iii) minimality. A set of matchings V satisfies horizon-L deterrence of external deviations if all possible deviations from any matching $\mu \in V$ to a matching outside V are deterred by a threat of ending worse off or equally well off. A set of matchings V satisfies horizon-L external stability if from any matching outside of V, there is a sequence of farsighted improving paths of length smaller than or equal to L leading to some matching in V. For $L \geq 1$, a set of matchings $V \subseteq \mathcal{M}$ is a horizon-L farsighted set if it is a minimal set satisfying horizon-L deterrence of external deviations and horizon-L external stability. From Herings et al. (2019), we have that a horizon-L farsighted set of matchings exists.

Theorem 2. *Let $\langle I, S, P, F \rangle$ be a priority-based matching problem and μ^T be the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is a horizon-L farsighted set for $L \geq 3\gamma$.*

Thus, our main result is robust to an alternative notion of limited farsightedness. The only difference is that one more degree of farsightedness is needed for deterring deviations from the TTC matching μ^T.

4 | Matching With Couples

Suppose that the set of agents is enlarged to $2n$ agents, $I = \{i_1, ..., i_{2n}\}$, and is partitioned into a set of couples $C = \{c_1, ..., c_n\} = \{(i_1, i_{n+1}), ..., (i_n, i_{2n})\}$. Let c be a generic couple. The partner of agent $i \in I$ is denoted $c(i)$. Let $S = \{s_1, ..., s_m\}$ be the set of objects. Let $P^\star = \{P_{c_1}, ..., P_{c_n}\}$ be the preference profile of the couples, where P_{c_l} is the strict preference of couple $c_l = (i_l, i_{n+l})$ over their objects and their outside options, $l = 1, ..., n$. The preference P_{c_l} of couple c_l is a linear order over $(S \cup i_l \times S \cup i_{n+l}) \setminus \{(s, s) | s \in S\}$. For instance, $P_{c_l} = (s_1, s_3), (s_2, s_4), (s_3, i_{n+l}), ..., (i_l, i_{n+l})$ indicates that couple $c_l = (i_l, i_{n+l})$ prefers i_l and i_{n+l} being matched to s_1 and s_3, respectively, to being matched to s_2 and s_4, respectively, and that the worst outcome for the couple is to be both unassigned. The pair of objects (s, s') are acceptable for couple c_l if $(s, s') P_{c_l} (i_l, i_{n+l})$. Let R_{c_l} be the weak preference relation associated with the strict preference relation P_{c_l}.

Definition 4. Let $\langle I, S, P^\star, F \rangle$ be a couples priority-based matching problem. A horizon-k improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, ..., \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, ..., L-1\}$, either

i. $\mu_{l+1} = \mu_l - (i, s) + (i, i)$ for some $(i, s) \in I \times S$ such that

$$(\mu_{\min\{l+k,L\}}(i), \mu_{\min\{l+k,L\}}(c(i))) \; P_{(i,c(i))}(\mu_l(i), \mu_l(c(i))),$$

or

ii. $\mu_{l+1} = \mu_l - (i, s) - (j, s') + (i, i) + (j, j)$ for some $(i, s), (j, s') \in I \times S$ such that

$$\begin{cases} (\mu_{\min\{l+k,L\}}(i), \mu_{\min\{l+k,L\}}(c(i))) P_{(i,c(i))}(\mu_l(i), \mu_l(c(i))) \\ (\mu_{\min\{l+k,L\}}(j), \mu_{\min\{l+k,L\}}(c(j))) \\ \quad R_{(j,c(j))}(\mu_l(j), \mu_l(c(j))), \end{cases}$$

or

iii. $\mu_{l+1} = \mu_l + (i, s) - (i, \mu_l(i)) - \{(j, s) | \mu_l(j) = s\} + \{(j, j) | \mu_l(j) = s\}$ for some $(i, s) \in I \times S$ such that $(\mu_{\min\{l+k,L\}}(i), \mu_{\min\{l+k,L\}}(c(i))) P_{(i,c(i))}(\mu_l(i), \mu_l(c(i)))$ and $F_s(i) < F_s(j)$ if $\mu_l(s) = j$.[23]

In the case of matching problems with couples, we require that, along a horizon-k improving path, each time some agent i is on the move, her couple $(i, c(i))$ is better off at the matches that they will get k steps ahead on the sequence compared to their current matches, except if the end matching of the sequence lies within her horizon. Another difference with respect to Definition 1 is that, as we deal with couples, we allow a pair of agents to simultaneously leave their objects (condition (ii) in Definition 4). The set of matchings $\mu' \in \mathcal{M}$ such that there is a horizon-k improving path from μ to μ' is denoted by $\bar{\phi}_k(\mu)$, so $\bar{\phi}_k(\mu) = \{\mu' \in \mathcal{M} | \mu \to_k \mu'\}$. Replacing $\phi_k(\mu)$ by $\bar{\phi}_k(\mu)$ in Definition 2, we obtain the definition of a horizon-k vNM stable set for priority-based matching problems with couples.

Example 3. (Klaus and Klijn 2007) Consider a couples priority-based matching problem $\langle I, S, P^\star, F \rangle$ with $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2, s_3\}$. Couples' preferences and objects' priorities are as follows.

Couples

$P_{(i_1,i_3)}$	$P_{(i_2,i_4)}$
(s_3, s_1)	(s_2, s_3)
(s_2, s_3)	
(s_1, s_2)	

Objects

F_{s_1}	F_{s_2}	F_{s_3}
i_3	i_3	i_3
i_1	i_2	i_4
	i_1	i_1

When couples are myopic, we have that $\mu'' = \{(i_1, s_2), (i_2, i_2), (i_3, s_3), (i_4, i_4)\}$ is the unique stable matching in Example 3. When couples and objects are farsighted, Atay et al. (2025) show that $\{\mu''\}$ is the unique vNM farsighted stable set. However, this matching μ'' is Pareto-dominated for agents (couples) by $\mu' = \{(i_1, s_3), (i_2, i_2), (i_3, s_1), (i_4, i_4)\}$. Notice that agent i_3 has priority for all three objects. When only agents are farsighted, can we stabilize some matching where the couple of i_3 gets their best possible match (i.e., (s_3, s_1))? If yes, how much farsightedness do we need?

In Example 3, the singleton set $\{\mu'\}$ is a horizon-k vNM stable set for $k \geq 3$ as there is a horizon-k improving path from any other matching to μ'. From any $\mu \neq \mu'$, looking k steps ahead, agent i_3 matches successively to each object that was assigned to another agent in μ. Next, agent i_3 followed by agent i_1 match to s_1 and s_3, respectively. They thereby reach μ' where the couple obtains their most preferred objects. In addition, the Pareto-dominated matching μ'' is now destabilized. From μ', the couple (i_1, i_3) will never engage a move toward μ'', as they strictly prefer μ' to μ'', and the other couple is indifferent between both matchings.[24]

Example 4. (Roth 2008) Consider a couples matching problem $\langle I, S, P^\star, F \rangle$ with $I = \{i_1, i_2, i_3, i_4\}$ and $S = \{s_1, s_2\}$. Couples' preferences and objects' priorities are as follows.

Couples

$P_{(i_1,i_3)}$	$P_{(i_2,i_4)}$
(s_1, s_2)	(s_1, i_4)
	(s_2, i_4)

Objects

F_{s_1}	F_{s_2}
i_1	i_1
i_2	i_2
i_3	i_3

Although there is again an agent who has priority over all other agents for both objects, Roth (2008) shows that there does not exist a stable matching in Example 4.[25] Furthermore, Atay et al. (2025) show that, when agents and objects behave farsightedly, there are no vNM farsighted stable sets in this example. However, when only one side is farsighted, the matchings $\mu' = \{(i_1, s_1), (i_2, i_2), (i_3, s_2), (i_4, i_4)\}$ and $\mu'' = \{(i_1, i_1), (i_2, s_1), (i_3, i_3), (i_4, i_4)\}$ are singleton horizon-k vNM stable sets for $k \geq 3$. Thus, farsightedness on behalf of the couples may help to stabilize Pareto-efficient outcomes.

Proposition 1. *Let $\langle I, S, P^\star, F\rangle$ be a couples priority-based matching problem. Suppose there is some agent $i^\star \in I$ such that $F_{i^\star}(s) < F_j(s)$ for all $s \in S, j \in I$ $(j \neq i^\star)$, and $m \geq 2n$. Each singleton set $\{\mu^\star\}$ where $\mu^\star$ is Pareto-efficient and $(\mu^\star(i^\star), \mu^\star(c(i^\star)))R_{(i^\star, c(i^\star))}(s, s')$ for all $s, s' \in S, s \neq s'$, is a horizon-k vNM stable set for all $k \geq 2n + 2$.*

This proposition tells us that, if there is some agent $i^\star$ who has priority over all objects and k is greater than the number of agents, then each Pareto-efficient matching where the couple of $i^\star$ gets their most preferred matches is a singleton horizon-k vNM stable set.

5 | Agents Own the Objects

Closely related to priority-based matching problems are matching problems where the agents own the object. Let $\langle I, S, P, F\rangle$ be a matching problem where each agent i owns an object s. The strict priority structure F of the objects over the agents is such that the priority F_s of object s only ranks the owner of object s. Without loss of generality, let agent i_l be the owner of object s_l, for $l = 1, \ldots, n$. Let i_s be a generic agent who owns object s.

Example 5. Consider a matching problem $\langle I, S, P, F\rangle$ with $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2, s_3\}$, and where agent i_l owns object s_l, for $l = 1, 2, 3$. Agents' preferences and endowments are as follows.

Agents			
Endowment	s_1	s_2	s_3
	P_{i_1}	P_{i_2}	P_{i_3}
	s_3	s_3	s_2
	s_1	s_2	s_3
	s_2	s_1	s_1

Objects		
F_{s_1}	F_{s_2}	F_{s_3}
i_1	i_2	i_3

In Example 5, $\mu^T = \{(i_1, s_1), (i_2, s_3), (i_3, s_2)\}$ is the matching obtained from the TTC algorithm. In the first round of the TTC algorithm, there is one cycle where agent i_2 points to object s_3, object s_3 points its owner i_3, agent i_3 points to object s_2, and object s_2 points its owner i_2. That is, $C_1 = \{c_1^1\}$ with $c_1^1 = (s_3, i_3, s_2, i_2)$. Agent i_2 is assigned to object s_3 and agent i_3 is assigned to object s_2: $m_1^1 = \{(i_2, s_3), (i_3, s_2)\}$, and so i_2 and i_3 exchange their objects. In the second round of the TTC algorithm, there is only one leftover agent, i_1, who points to object s_1 that she owns and one leftover object, s_1, that points to its owner i_1. That is, $C_2 = \{c_2^1\}$ with $c_2^1 = (s_1, i_1)$. Agent i_1 is assigned to her own object s_1: $m_2^1 = \{(i_1, s_1)\}$, and so $\mu^T = m_1^1 \cup m_2^1$.

Roth and Postlewaite (1977) show that, for any matching problem $\langle I, S, P, F\rangle$ where each agent i owns an object s, there is always a unique matching that is in the core. Moreover, this matching can be obtained with the TTC algorithm.[26]

Definition 5. Let $\langle I, S, P, F\rangle$ be a matching problem where each agent i owns an object s. A horizon-k improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M}\backslash\{\mu\}$ is a finite sequence of distinct matchings $\mu_0, \ldots, \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, \ldots, L-1\}$ either

- **i.** $\mu_{l+1} = \mu_l - (i, s) + (i, i)$ for some $(i, s) \in I \times S$ such that $\mu_{\min\{l+k, L\}}(i) P_i \mu_l(i)$, or
- **ii.** $\mu_{l+1} = \mu_l + (i, s) - (i, \mu_l(i)) - \{(j, s) | \mu_l(j) = s\} + \{(j, j) \mid \mu_l(j) = s\}$ for some $(i, s) \in I \times S$ such that $\mu_{\min\{l+k, L\}}(i) P_i \mu_l(i)$ and $\mu_{\min\{l+k, L\}}(i_s) P_{i_s} \mu_l(i_s)$.

In the case of matching problems where each agent owns an object, we still require that, along a horizon-k improving path, each time some agent i is on the move, she is better off at the match she will get k steps ahead on the sequence compared to her current match. Moreover, if agent i matches to s, we also require that the owner of the object (i.e., i_s) prefers the match he will get k steps ahead compared to his current match. In other words, the owner of the object has a word to say about the assignment of his endowment to some agent.[27]

The set of matchings $\mu' \in \mathcal{M}$ such that there is a horizon-k improving path from μ to μ' is denoted by $\tilde{\phi}_k(\mu)$, so $\tilde{\phi}_k(\mu) = \{\mu' \in \mathcal{M} | \mu \rightarrow_k \mu'\}$. Replacing $\phi_k(\mu)$ by $\tilde{\phi}_k(\mu)$ in Definition 2, we obtain the definition of a horizon-k vNM stable set for matching problems where each agent owns an object.

Theorem 3. *Let $\langle I, S, P, F\rangle$ be a matching problem where each agent i owns an object s and μ^T is the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is the unique horizon-k vNM stable set for $k \geq 3\gamma - 1$.*

Corollary 2. *Let $\langle I, S, P, F\rangle$ be a matching problem where each agent i owns an object s and μ^T is the matching obtained from the TTC mechanism. The singleton set $\{\mu^T\}$ is the unique vNM farsighted stable set.*[28]

6 | Conclusion

We have considered one-to-one priority-based matching problems with limited farsightedness. We have shown that, once agents are sufficiently farsighted, the matching obtained from the TTC algorithm becomes stable: a singleton set consisting of the TTC matching is a horizon-k vNM stable set if the degree of farsightedness is greater than three times the number of agents in the largest cycle of the TTC. On the contrary, the matching obtained from the DA algorithm may not belong to any horizon-k vNM stable set for k large enough. Hence, the TTC mechanism satisfies Pareto efficiency, strategy-proofness, and (limited) farsighted stability. Notice that a mechanism is strategy-proof if no agent has incentives to misrepresent her preferences anticipating perfectly the outcome of the TTC algorithm. Therefore, strategy-proofness implicitly presumes some degree of farsightedness on behalf of the agents. Thus, it seems more consistent to look for a mechanism that satisfies strategy-proofness together with (limited) farsighted stability.

Our main results are robust to alternative notions of limited farsightedness. However, they do not hold per se for many-to-one priority-based matching problems. More coordination and cooperation on behalf of the agents are required to sustain the TTC matching as a singleton horizon-k vNM stable set: one needs to allow group of agents to move all together. In matching markets with couples, farsightedness may improve both efficiency and stability. When each agent owns an object, a singleton set consisting of the TTC matching is the unique horizon-k vNM stable set.

For future research, it would be interesting to investigate whether soulmate mechanisms such as the iterated matching of soulmates algorithm (see Leo et al. 2021) do satisfy farsighted stability.[29]

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Data Availability Statement

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Endnotes

[1] Roth and Sotomayor (1990) and Haeringer (2017) provide a general introduction to matching problems.

[2] Abdulkadiroğlu and Sönmez (2003) show that both mechanisms are strategy-proof: truthful preference revelation is a weakly dominant strategy for the agents. Dubins and Freedman (1981) and Roth (1982) were the first to show that the DA mechanism satisfies strategy-proofness in one-to-one matching problems.

[3] Kesten (2010) introduces the Efficiency-Adjusted Deferred Acceptance (EADA) mechanism to improve efficiency upon the DA mechanism. Reny (2022) provides the Priority-Efficient (PE) mechanism that always selects a Pareto-efficient matching that dominates the DA stable matching, but the PE mechanism is not strategy-proof.

[4] See, e.g., Kirchsteiger et al. (2016), Teteryatnikova and Tremewan (2020).

[5] See Chwe (1994), Page and Wooders (2009), Mauleon et al. (2011), Ray and Vohra (2015, 2019), Herings et al. (2019, 2020), and Luo et al. (2021) for definitions of the farsighted stable set. Alternative notions of farsightedness are proposed by Page et al. (2005) and Grandjean et al. (2011), among others.

[6] In some sense, agents behave strategically by anticipating further moves while objects behave mechanically by accepting or rejecting agents based on their priority lists.

[7] Morrill (2015) and Hakimov and Kesten (2018) propose variations of the TTC for school choice problems. Atay et al. (2025) show that the matchings obtained from the variations are all farsightedly stable. For one-to-one priority-based problems, all variations coincide.

[8] See Klaus and Klijn (2005, 2007), Roth (2008). Biró and Klijn (2013) provide a detailed overview of matching markets with couples under preferences.

[9] Throughout the paper, we use the notation $\subseteq$ for weak inclusion and $\subset$ for strict inclusion. Finally, # will refer to the notion of cardinality.

[10] A mechanism is individually rational (non-wasteful/stable/Pareto-efficient) if it always selects an individually rational (non-wasteful/stable/Pareto-efficient) matching. A mechanism is strategy-proof if no agent can ever benefit by unilaterally misrepresenting her preferences.

[11] Abdulkadiroğlu et al. (2020) show that the TTC mechanism is justified envy minimal in the class of Pareto-efficient and strategy-proof mechanisms. In addition, Doğan and Ehlers (2022) find that, for any stability comparison satisfying three basic properties, the TTC mechanism is minimally unstable among Pareto-efficient and strategy-proof mechanisms.

[12] Doğan and Ehlers (2021) study efficient and minimally unstable Pareto improvements over the DA mechanism. Che and Tercieux (2019) show that both Pareto efficiency and stability can be achieved asymptotically using DA and TTC mechanisms when agents have uncorrelated preferences. Kesten (2006) compares the TTC and DA mechanisms with respect to some additional properties (fairness, resource monotonicity, population monotonicity, and consistency).

[13] We use the notation + for adding pairs and $^{-}$ for deleting pairs.

[14] Ehlers (2007) and Herings et al. (2017) study vNM stable sets when all agents are myopic in two-sided matching problems.

[15] In all examples, we only provide the agents' preferences over objects that are acceptable for them (i.e., those objects that are strictly preferred over their outside options).

[16] Hence, it is already sufficient that agent i_3 looks forward 4 (instead of 5) steps ahead for having incentives to engage her first move toward μ_4 from μ_0. Agents may even coordinate on shorter horizon-k improving paths. For instance, by replacing μ_2 by $\mu_2' = \{(i_1, i_1), (i_2, i_2), (i_3, s_1)\}$ and μ_3 by $\mu_3' = \{(i_1, i_1), (i_2, i_2), (i_3, s_2)\}$, agents i_2 and i_3 only need to look at least three steps ahead for moving along the improving path toward μ_4. See Remark 2.

[17] Proposition 3 in Luo et al. (2021) provides a characterization of a horizon-1 vNM stable set of networks. As matchings are a subclass of networks and the DA matching is stable, it follows that the DA matching belongs to all horizon-1 vNM stable sets.

[18] Mauleon et al. (2011) define and characterize the vNM farsighted stable set for two-sided matching problems.

[19] For one-to-one matching problems, Herings et al. (2025) show that, if the woman-optimal stable matching is dominated from the woman point of view by an individually rational matching, then no core element can be part of a myopic-farsighted stable set.

[20] The object s' is said to be a copy of s if and only if $F_s = F_{s'}$ and all agents are indifferent between s and s'.

[21] Depending in which order the agents point to each copy leads to different but equivalent TTC matchings. In fact, all TTC matchings are identical up to a permutation of the copies. Each agent gets the same object or one of its copies.

[22]Doğan and Ehlers (2025) show that when one side is farsighted and the other side is myopic, then neither the EADA matching nor the TTC matching is necessarily a singleton myopic-farsighted stable set in many-to-one matching problems with coalitional or pairwise deviations.

[23]By allowing for a pair of agents to match simultaneously to some objects, Proposition 1 still holds but less farsightedness on behalf of the agents may be necessary.

[24]In Example 3, there is another horizon-k vNM stable set $\{\mu'''\}$ where the other couple (i_2, i_4) gets their preferred assignment: $\mu''' = \{(i_1, i_1), (i_2, s_2), (i_3, i_3), (i_4, s_3)\}$.

[25]Klaus and Klijn (2005) look for restrictions on the preferences to guarantee the existence of a stable matching in markets with couples.

[26]For matching problems with private endowments, Ma (1994) shows that a mechanism is strategy-proof, Pareto-efficient, and individually rational if and only if it uses the TTC algorithm.

[27]The notion of contractual stability captures a similar idea by requiring the consent of coalition partners. See, e.g., Diamantoudi and Xue (2003), Caulier et al. (2013), Caulier et al. (2013), and Mauleon et al. (2016).

[28]In an exchange economy with indivisible goods of Shapley and Scarf (1974), Kawasaki (2010), and Klaus et al. (2010) show that there exists a unique vNM farsighted stable set, which coincides with the set of competitive allocations. Thus, they obtain a similar result to Corollary 2, except that they allow for coalitional moves while agents can only move one at a time in our definition of vNM farsighted stable set.

[29]For roommate markets, Klaus et al. (2011) study farsighted stability, whereas Atay et al. (2021) look at the credibility of blocking pairs.

[30]In an older version, we provided a tighter bound equal to $2\gamma + 1$. We thank an anonymous reviewer for suggesting an even lower tighter bound equal to $\gamma + 2$.

[31]From Lemma 2 in Herings et al. (2019), we have that for every $L \geq 1$, for every $\mu \in \mathcal{M}$, it holds that $\widehat{\phi}^{\infty}_{L}(\mu) \subseteq \widehat{\phi}^{\infty}_{L+1}(\mu)$, and that for $L \geq k^{\star}$, for every $\mu \in \mathcal{M}$, it holds that $\widehat{\phi}^{\infty}_{L}(\mu) = \widehat{\phi}^{\infty}_{L+1}(\mu) = \widehat{\phi}^{\infty}_{\infty}(\mu)$.

[32]We use the notational convention that $\widehat{\phi}_{-1}(\mu) = \emptyset$ for every $\mu \in \mathcal{M}$.

[33]Notice that along the horizon-k improving path from the proof of Theorem 1, an agent matches either to an unassigned object whose owner is unmatched or to an object that she is the owner. Hence, the owner of the object does not block her move toward the TTC matching μ^T.

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Appendix A

Proof of Theorem 1

Proof. As $\{\mu^T\}$ is a singleton set, internal stability (IS) is satisfied. (ES) Take any matching $\mu \neq \mu^T$; we need to show that $\phi_k(\mu) \ni \mu^T$ for $k \geq (3\gamma - 1)$. We build in steps a horizon-k improving path from μ to μ^T for $k \geq (3\gamma - 1)$. Remember that $\mu^T = \bigcup_{k=1}^{\bar{k}} M_k$ where $M_k = \bigcup_{l=1}^{L_k} m_k^l$ are all the matches between agents and objects formed in Step k of the TTC algorithm, and m_k^l is given by Expressions (1) and (2).

Step 1.1. If $m_1^1 \subseteq \mu$ and $1 \neq L_1$ then go to Step 1.2 with $\mu'''_{1,1} = \mu$. If $m_1^1 \subseteq \mu$ and $1 = L_1$ then go to Step 1. End with $\mu'''_{1,L_1} = \mu$. If $m_1^1 \nsubseteq \mu$ then go to Step 1.1.A.

Step 1.1.A. If $\{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} \subseteq \mu$ then go to Step 1.1.B with $\mu'_{1,1} = \mu$. If $\{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} \nsubseteq \mu$ then there is some agent i such that $s \neq \mu(i) \neq \mu^T(i)$ and $s \mapsto i$ with $i, s \in c_1^1$. This agent i matches with object s that ranks her first on its priority list. We reach the matching $\mu'_{1,1,1} = \mu + (i,s) - (i, \mu(i)) - \{(j,s) | \mu(j) = s\} + \{(j,j) | \mu(j) = s\}$ where $s \mapsto i$ and $i, s \in c_1^1$. If $\{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} \subseteq \mu'_{1,1,1}$ then go to Step 1.1.B with $\mu'_{1,1} = \mu'_{1,1,1}$. If $\{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} \nsubseteq \mu'_{1,1,1}$, then there is some agent i' such that $s' \neq \mu'_{1,1,1}(i') \neq \mu^T(i')$ and $s' \mapsto i'$ with $i', s' \in c_1^1$. This agent i' matches with object s' that ranks her first on its priority list. We reach the matching $\mu'_{1,1,2} = \mu'_{1,1,1} + (i', s') - (i', \mu'_{1,1,1}(i')) - \{(j, s') | \mu'_{1,1,1}(j) = s'\} + \{(j,j) | \mu'_{1,1,1}(j) = s'\}$, where $s' \mapsto i'$ and $i', s' \in c_1^1$. We proceed as above until we reach the matching $\mu'_{1,1} = \mu + \{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} - \{(i, \mu(i)) | i, s \in c_1^1 \text{ and } s \mapsto i\} - \{(j,s) | j \notin c_1^1, i, s \in c_1^1, s \mapsto i \text{ and } \mu(s) = j\} + \{(j,j) | j \notin c_1^1, i, s \in c_1^1, s \mapsto i \text{ and } \mu(s) = j\}$, where each agent involved in c_1^1 is matched to the object that ranks her first on its priority list. Step 1.1.A counts at most $\bar{c}_1^1$ steps.

Step 1.1.B. Let $\mathcal{I}_1^1 = \{(i^r)\}_{r=1}^{\bar{c}_1^1}$ be such that $i^r \in c_1^1$ and $i^r = i_{o_r} \neq i^{r+1} = i_{o_{r+1}}$ with $o_r < o_{r+1}$ for $r = 1, \ldots, \bar{c}_1^1 - 1$. That is, $\mathcal{I}_1^1$ is an ordered set of the agents involved in cycle c_1^1, where $\bar{c}_1^1 = \#\{i \in I | i \in c_1^1\}$ is the number of agents involved in cycle c_1^1. From the matching $\mu'_{1,1}$, agents i^1 to $i^{\bar{c}_1^1 - 1}$ successively leave their objects to reach the matching $\mu''_{1,1} = \mu'_{1,1} - \{(i,s) | i, s \in c_1^1, s \mapsto i \text{ and } i \neq i^{\bar{c}_1^1}\} + \{(i,i) | i, s \in c_1^1, s \mapsto i \text{ and } i \neq i^{\bar{c}_1^1}\}$, where only agent $i^{\bar{c}_1^1}$ is still matched to the object that ranks her first on its priority list. Step 1.1.B counts at most $\bar{c}_1^1 - 1$ steps.

Step 1.1.C. From the matching $\mu''_{1,1}$, agent $i^{\bar{c}_1^1}$ first matches with her top choice object s to reach the matching $\mu''_{1,1} + (i^{\bar{c}_1^1}, s) - (i^{\bar{c}_1^1}, \mu''_{1,1}(i^{\bar{c}_1^1}))$ where s is such that $i \mapsto s$ and $i, s \in c_1^1$. Notice that s was unassigned at $\mu''_{1,1}$, whereas the object that ranks $i^{\bar{c}_1^1}$ first on its priority list, i.e., $\mu''_{1,1}(i^{\bar{c}_1^1})$, is now unassigned. Next, agents i^1 to $i^{\bar{c}_1^1 - 1}$ successively match to their top choice object to reach the matching $\mu'''_{1,1} = \mu'_{1,1} - \{(i,s) | i, s \in c_1^1 \text{ and } s \mapsto i\} + \{(i,s) | i, s \in c_1^1 \text{ and } i \mapsto s\}$. Step 1.1.C counts at most $\bar{c}_1^1$ steps. We have reached $\mu'''_{1,1}$ with $m_1^1 \subseteq \mu'''_{1,1}$ and so agents belonging to c_1^1 are assigned to the same object as in μ^T and they obtain their best possible match. Step 1.1 counts at most $3\bar{c}_1^1 - 1$ steps. Hence, it is sufficient that the agents who move in Step 1.1 look forward $3\bar{c}_1^1 - 1$ steps ahead to have incentives for engaging the move toward the matching $\mu'''_{1,1}$, where they already get the object assigned by the TTC. Once they reach those matches, they do not move afterwards. If $1 \neq L_1$, then go to Step 1.2. Otherwise, go to Step 1. End with $\mu'''_{1,L_1} = \mu'''_{1,1}$.

Step 1.*l*. ($l > 1$) If $m_1^l \subseteq \mu'''_{1,l-1}$ and $l \neq L_1$ then go to Step 1.l+1 with $\mu'''_{1,l} = \mu'''_{1,l-1}$. If $m_1^l \subseteq \mu'''_{1,l-1}$ and $l = L_1$ then go to Step 1. End with $\mu'''_{1,L_1} = \mu'''_{1,l-1}$. If $m_1^l \nsubseteq \mu'''_{1,l-1}$ then go to Step 1.*l*.A.

Step 1.*l*.A. If $\{(i,s) | i, s \in c_1^l \text{ and } s \mapsto i\} \subseteq \mu'''_{1,l-1}$ then go to Step 1.*l*.B with $\mu'_{1,l} = \mu'''_{1,l-1}$. If $\{(i,s) | i, s \in c_1^l \text{ and } s \mapsto i\} \nsubseteq \mu'''_{1,l-1}$ then there is some agent i such that $s \neq \mu'''_{1,l-1}(i) \neq \mu^T(i)$ and $s \mapsto i$ with $i, s \in c_1^l$. This agent i matches with object s that ranks her first on its priority list. We reach the matching $\mu'_{1,l,1} = \mu'''_{1,l-1} + (i,s) - (i, \mu'''_{1,l-1}(i)) - \{(j,s) | \mu'''_{1,l-1}(j) = s\} + \{(j,j) | \mu'''_{1,l-1}(j) = s\}$ where $s \mapsto i$

and $i, s \in c_1^l$. If $\{(i,s)|i, s \in c_1^l \text{ and } s \mapsto i\} \subseteq \mu'_{1,l,1}$ then go to Step 1l.B with $\mu'_{1,l} = \mu'_{1,l,1}$. If $\{(i,s)|i, s \in c_1^l$ and $s \mapsto i\} \not\subseteq \mu'_{1,l,1}$, then there is some agent i' such that $s' \neq \mu'_{1,l,1}(i') \neq \mu^T(i')$ and $s' \mapsto i'$ with $i', s' \in c_1^l$. This agent i' matches with object s' that ranks her first on its priority list. We reach the matching $\mu'_{1,l,2} = \mu'_{1,l,1} + (i', s') - (i', \mu'_{1,l,1}(i')) - \{(j, s')|\mu'_{1,l,1}(j) = s'\} + \{(j,j)|\mu'_{1,l,1}(j) = s'\}$, where $s' \mapsto i'$ and $i', s' \in c_1^l$. We proceed as above until we reach the matching $\mu'_{1,l} = \mu'''_{1,l-1} + \{(i,s)|i, s \in c_1^l \text{ and } s \mapsto i\} - \{(i, \mu'''_{1,l-1}(i))|i, s \in c_1^l \text{ and } s \mapsto i\} - \{(j,s)|j \notin c_1^l, i, s \in c_1^l, s \mapsto i \text{ and } \mu'''_{1,l-1}(s) = j\} + \{(j,j)|j \notin c_1^l, i, s \in c_1^l, s \mapsto i \text{ and } \mu'''_{1,l-1}(s) = j\}$, where each agent involved in c_1^l is matched to the object that ranks her first on its priority list. Step 1l.A counts at most $\bar{c}_1^l$ steps.

Step 1l.B. Let $\mathcal{I}_1^l = \{(i^r)\}_{r=1}^{\bar{c}_1^l}$ be such that $i^r \in c_1^l$ and $i^r = i_{o_r} \neq i^{r+1} = i_{o_{r+1}}$ with $o_r < o_{r+1}$ for $r = 1, \ldots, \bar{c}_1^l - 1$. That is, $\mathcal{I}_1^l$ is an ordered set of the agents involved in cycle c_1^l, where $\bar{c}_1^l = \#\{i \in I | i \in c_1^l\}$ is the number of agents involved in cycle c_1^l. From the matching $\mu'_{1,l}$, agents i^1 to $i^{\bar{c}_1^l - 1}$ successively leave their objects to reach the matching $\mu''_{1,l} = \mu'_{1,l} - \{(i,s) \mid i, s \in c_1^l, s \mapsto i \text{ and } i \neq i^{\bar{c}_1^l}\} + \{(i,i)|i, s \in c_1^l, s \mapsto i \text{ and } i \neq i^{\bar{c}_1^l}\}$, where only agent $i^{\bar{c}_1^l}$ is still matched to the object that ranks her first on its priority list. Step 1l.B counts at most $\bar{c}_1^l - 1$ steps.

Step 1l.C. From the matching $\mu''_{1,l}$, agent $i^{\bar{c}_1^l}$ first matches with her top choice object s to reach the matching $\mu''_{1,l} + (i^{\bar{c}_1^l}, s) - (i^{\bar{c}_1^l}, \mu''_{1,l}(i^{\bar{c}_1^l}))$, where s is such that $i \mapsto s$ and $i, s \in c_1^l$. Notice that s was unassigned at $\mu''_{1,l}$, whereas the object that ranks $i^{\bar{c}_1^l}$ first on its priority list, i.e., $\mu''_{1,l}(i^{\bar{c}_1^l})$, is now unassigned. Next, agents i^1 to $i^{\bar{c}_1^l - 1}$ successively match to their top choice object to reach the matching $\mu'''_{1,l} = \mu''_{1,l} - \{(i,s)|i, s \in c_1^l \text{ and } s \mapsto i\} + \{(i,s)|i, s \in c_1^l \text{ and } i \mapsto s\}$. Step 1$l$.C counts at most $\bar{c}_1^l$ steps. We have reached $\mu'''_{1,l}$ with $m_1^l \subseteq \mu'''_{1,l}$ and so agents belonging to c_1^l are assigned to the same object as in μ^T. Step 1l counts at most $3\bar{c}_1^l - 1$ steps. Hence, it is sufficient that the agents who move in Step 1l look forward $3\bar{c}_1^l - 1$ steps ahead to have incentives for engaging the move toward the matching $\mu'''_{1,l}$, where they already get the object assigned by the TTC. Once they reach those matches, they do not move afterwards. If $1 \neq L_1$, then go to Step 1l + 1. Otherwise, go to Step 1. End with $\mu'''_{1,L_1} = \mu'''_{1,l}$.

Step 1.End. We have reached μ'''_{1,L_1}, with $\cup_{l=1}^{L_1} m_1^l = M_1 \subseteq \mu'''_{1,L_1}$. If $\mu'''_{1,L_1} = \mu^T$ then the process ends. Otherwise, go to Step 2.1.

Step k.1. ($k \geq 2$) If $m_k^1 \subseteq \mu'''_{k-1,L_{k-1}}$ and $1 \neq L_k$ then go to Step k.2 with $\mu'''_{k,1} = \mu'''_{k-1,L_{k-1}}$. If $m_k^1 \subseteq \mu'''_{k-1,L_{k-1}}$ and $1 = L_k$ then go to Step k. End with $\mu'''_{k,L_k} = \mu'''_{k-1,L_{k-1}}$. If $m_k^1 \not\subseteq \mu'''_{k-1,L_{k-1}}$ then go to Step k.1.A.

Step k.1.A. If $\{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} \subseteq \mu'''_{k-1,L_{k-1}}$ then go to Step k.1.B with $\mu'_{k,1} = \mu'''_{k-1,L_{k-1}}$. If $\{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} \not\subseteq \mu'''_{k-1,L_{k-1}}$ then there is some agent i such that $s \neq \mu'''_{k-1,L_{k-1}}(i) \neq \mu^T(i)$ and $s \mapsto i$ with $i, s \in c_k^1$. This agent i matches with object $s \in S \backslash \bigcup_{l=1}^{k-1} S_l$ that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. We reach the matching $\mu'_{k,1,1} = \mu'''_{k-1,L_{k-1}} + (i,s) - (i, \mu'''_{k-1,L_{k-1}}(i)) - \{(j,s)|\mu'''_{k-1,L_{k-1}}(j) = s\} + \{(j,j)|\mu'''_{k-1,L_{k-1}}(j) = s\}$, where $s \mapsto i$ and $i, s \in c_k^1$. If $\{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} \subseteq \mu'_{k,1,1}$ then go to Step k.1.B with $\mu'_{k,1} = \mu'_{k,1,1}$. If $\{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} \not\subseteq \mu'_{k,1,1}$ then there is some agent i' such that $s' \neq \mu'_{k,1,1}(i') \neq \mu^T(i')$ and $s' \mapsto i'$ with $i', s' \in c_k^1$. This agent i' matches with object $s' \in S \backslash \bigcup_{l=1}^{k-1} S_l$ that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. We reach the matching $\mu'_{k,1,2} = \mu'_{k,1,1} + (i', s') - (i', \mu'_{k,1,1}(i')) - \{(j,s')|\mu'_{k,1,1}(j) = s'\} + \{(j,j)|\mu'_{k,1,1}(j) = s'\}$, where $s' \mapsto i'$ and $i', s' \in c_k^1$. We proceed as above until we reach the matching $\mu'_{k,1} = \mu'''_{k-1,L_{k-1}} + \{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} - \{(i, \mu'''_{k-1,L_{k-1}}(i))|i, s \in c_k^1 \text{ and } s \mapsto i\} - \{(j,s)| j \notin c_k^1, i, s \in c_k^1, s \mapsto i \text{ and } \mu'''_{k-1,L_{k-1}}(s) = j\} + \{(j,j)|j \notin c_k^1, i, s \in c_k^1, s \mapsto i \text{ and } \mu'''_{k-1,L_{k-1}}(s) = j\}$, where each agent involved in c_k^1 is matched to the object that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. Step k.1.A counts at most $\bar{c}_1^k$ steps.

Step k.1.B. Let $\mathcal{I}_k^1 = \{(i^r)\}_{r=1}^{\bar{c}_k^1}$ be such that $i^r \in c_k^1$ and $i^r = i_{o_r} \neq i^{r+1} = i_{o_{r+1}}$ with $o_r < o_{r+1}$ for $r = 1, \ldots, \bar{c}_k^1 - 1$. That is, $\mathcal{I}_k^1$ is an ordered set of the agents involved in cycle c_k^1, where $\bar{c}_k^1 = \#\{i \in I | i \in c_k^1\}$ is the number of agents involved in cycle c_k^1. From the matching $\mu'_{k,1}$, agents i^1 to $i^{\bar{c}_k^1 - 1}$ successively leave their objects to reach the matching $\mu''_{k,1} = \mu'_{k,1} - \{(i,s)| i, s \in c_k^1, s \mapsto i \text{ and } i \neq i^{\bar{c}_k^1}\} + \{(i,i)|i, s \in c_k^1, s \mapsto i \text{ and } i \neq i^{\bar{c}_k^1}\}$, where only agent $i^{\bar{c}_k^1}$ is still matched to the object that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. Step k.1.B counts at most $\bar{c}_k^k - 1$ steps.

Step k.1.C. From the matching $\mu''_{k,1}$, agent $i^{\bar{c}_k^1}$ first matches with her top choice object $s \in S \backslash \bigcup_{l=1}^{k-1} S_l$ to reach the matching $\mu''_{k,1} + (i^{\bar{c}_k^1}, s) - (i^{\bar{c}_k^1}, \mu''_{k,1}(i^{\bar{c}_k^1}))$, where s is such that $i \mapsto s$ and $i, s \in c_1^k$. Notice that s was unassigned at $\mu''_{k,1}$, whereas the object that ranks $i^{\bar{c}_k^1}$ first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$, i.e., $\mu''_{k,1}(i^{\bar{c}_k^1})$, is now unassigned. Next, agents i^1 to $i^{\bar{c}_k^1 - 1}$ successively match to their top choice object in $S \backslash \bigcup_{l=1}^{k-1} S_l$ to reach the matching $\mu'''_{k,1} = \mu'_{k,1} - \{(i,s)|i, s \in c_k^1 \text{ and } s \mapsto i\} + \{(i,s)|i, s \in c_k^1 \text{ and } i \mapsto s\}$. Step 2.1.C counts at most $\bar{c}_k^1$ steps. We have reached $\mu'''_{k,1}$ with $m_k^1 \subseteq \mu'''_{k,1}$ and so agents belonging to c_k^1 are assigned to the same object as in μ^T. Step k.1 counts at most $3\bar{c}_k^1 - 1$ steps. Hence, it is sufficient that the agents who move in Step k.1 look forward $3\bar{c}_k^1 - 1$ steps ahead to have incentives for engaging the move toward the matching $\mu'''_{k,1}$, where they already get the object assigned by the TTC. Once they reach those matches they do not move afterwards. If $1 \neq L_k$, then go to Step k.2. Otherwise, go to Step k. End with $\mu'''_{k,L_k} = \mu'''_{k,1}$.

Step $k.l$. ($l > 1$) If $m_k^l \subseteq \mu'''_{k,l-1}$ and $l \neq L_k$ then go to Step $k.l$+1 with $\mu'''_{k,l} = \mu'''_{k,l-1}$. If $m_k^l \subseteq \mu'''_{k,l-1}$ and $l = L_k$ then go to Step k. End with $\mu'''_{k,L_k} = \mu'''_{k,l-1}$. If $m_k^l \not\subseteq \mu'''_{k,l-1}$ then go to Step $k.l$.A.

Step $k.l$.A. If $\{(i,s)|i, s \in c_k^l \text{ and } s \mapsto i\} \subseteq \mu'''_{k,l-1}$ then go to Step $k.l$.B with $\mu'_{k,l} = \mu'''_{k,l-1}$. If $\{(i,s)|i, s \in c_k^l \text{ and } s \mapsto i\} \not\subseteq \mu'''_{k,l-1}$ then there is some agent i such that $s \neq \mu'''_{k,l-1}(i) \neq \mu^T(i)$ and $s \mapsto i$ with $i, s \in c_k^l$. This agent i matches with object s that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. We reach the matching $\mu'_{k,l,1} = \mu'''_{k,l-1} + (i,s) - (i, \mu'''_{k,l-1}(i)) - \{(j,s)|\mu'''_{k,l-1}(j) = s\} + \{(j,j)|\mu'''_{k,l-1}(j) = s\}$, where $s \mapsto i$ and $i, s \in c_k^l$. If $\{(i,s)|i, s \in c_k^l \text{ and } s \mapsto i\} \subseteq \mu'_{k,l,1}$ then go to Step $k.l$.B with $\mu'_{k,l} = \mu'_{k,l,1}$. If $\{(i,s)|i, s \in c_k^l \text{ and } s \mapsto i\} \not\subseteq \mu'_{k,l,1}$ then there is some agent i' such that $s' \neq \mu'_{k,l,1}(i') \neq \mu^T(i')$ and $s' \mapsto i'$ with $i', s' \in c_k^l$. This agent i' matches with object s' that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. We reach the matching $\mu'_{k,l,2} = \mu'_{k,l,1} + (i', s') - (i', \mu'_{k,l,1}(i')) - \{(j,s')|\mu'_{k,l,1}(j) = s'\} + \{(j,j)| \mu'_{k,l,1}(j) = s'\}$, where $s' \mapsto i'$ and $i', s' \in c_k^l$. We proceed as above until we reach the matching $\mu'_{k,l} = \mu'''_{k,l-1} + \{(i,s)|i,s \in c_k^l \text{ and } s \mapsto i\} - \{(i,\mu'''_{k,l-1}(i))|i,s \in c_k^l \text{ and } s \mapsto i\} - \{(j,s)|j \notin c_k^l, i,s \in c_k^l, s \mapsto i \text{ and } \mu'''_{k,l-1}(s) = j\} + \{(j,j)|j \notin c_k^l, i,s \in c_k^l, s \mapsto i \text{ and } \mu'''_{k,l-1}(s) = j\}$, where each agent involved in c_k^l is matched to the object that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. Step 2.$l$. A counts at most $\bar{c}_k^l$ steps.

Step $k.l$.B. Let $\mathcal{I}_k^l = \{(i^r)\}_{r=1}^{\bar{c}_k^l}$ be such that $i^r \in c_k^l$ and $i^r = i_{o_r} \neq i^{r+1} = i_{o_{r+1}}$ with $o_r < o_{r+1}$ for $r = 1, ..., \bar{c}_k^l - 1$. That is, $\mathcal{I}_k^l$ is an ordered set of the agents involved in cycle c_k^l, where $\bar{c}_k^l = \#\{i \in I | i \in c_k^l\}$ is the number of agents involved in cycle c_k^l. From the matching $\mu'_{k,l}$, agents i^1 to $i^{\bar{c}_k^l - 1}$ successively leave their objects to reach the matching $\mu''_{k,l} = \mu'_{k,l} - \{(i,s)|\, i,s \in c_k^l, s \mapsto i \text{ and } i \neq i^{\bar{c}_k^l}\} + \{(i,i)|i,s \in c_k^l, s \mapsto i \text{ and } i \neq i^{\bar{c}_k^l}\}$, where only agent $i^{\bar{c}_k^l}$ is still matched to the object that ranks her first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$. Step $k.l$.B counts at most $\bar{c}_k^l - 1$ steps.

Step $k.l$.C. From the matching $\mu''_{k,l}$, agent $i^{\bar{c}_k^l}$ first matches with her top choice object $s \in S \backslash \bigcup_{l=1}^{k-1} S_l$ to reach the matching $\mu''_{k,l} + (i^{\bar{c}_k^l}, s) - (i^{\bar{c}_k^l}, \mu''_{k,l}(i^{\bar{c}_k^l}))$, where s is such that $i \mapsto s$ and $i, s \in c_k^l$. Notice that s was unassigned at $\mu''_{k,l}$, whereas the object that ranks $i^{\bar{c}_k^l}$ first on its priority list among agents belonging to $I \backslash \bigcup_{l=1}^{k-1} I_l$, i.e., $\mu''_{k,l}(i^{\bar{c}_k^l})$, is now unassigned. Next, agents i^1 to $i^{\bar{c}_k^l - 1}$ successively match to their top choice object in $S \backslash \bigcup_{l=1}^{k-1} S_l$ to reach the matching $\mu'''_{k,l} = \mu'_{k,l} - \{(i,s)|i,s \in c_k^l \text{ and } s \mapsto i\} + \{(i,s)|i,s \in c_k^l \text{ and } i \mapsto s\}$. Step $k.l$.C counts at most $\bar{c}_k^l$ steps. We have reached $\mu'''_{k,l}$ with $m_k^l \subseteq \mu'''_{k,l}$ and so agents belonging to c_k^l are assigned to the same object as in μ^T. Step $k.l$ counts at most $3\bar{c}_k^l - 1$ steps. Hence, it is sufficient that the agents who move in Step $k.l$ look forward $3\bar{c}_k^l - 1$ steps ahead to have incentives for engaging the move toward the matching $\mu'''_{k,l}$, where they already get the object assigned by the TTC. Once they reach those matches, they do not move afterwards. If $1 \neq L_k$, then go to Step $k.l + 1$. Otherwise, go to Step k. End with $\mu'''_{k,L_k} = \mu'''_{k,l}$.

Step k. End. We have reached μ'''_{k,L_k} with $\bigcup_{k'=1}^{k} M_{k'} \subseteq \mu'''_{k,L_k}$. If $\mu'''_{k,L_k} = \mu^T$ then the process ends. Otherwise, go to Step $k + 1$. 1.

End. The process goes on until Step $\bar{k}$, where we reach $\mu'''_{\bar{k},L_{\bar{k}}} = \bigcup_{k=1}^{\bar{k}} M_k = \mu^T$. Given $k \geq 3\gamma - 1$, we have that, along the constructed horizon-k improving path, each time an agent i is on the move, she has incentives to do so, as her end match (i.e., her TTC match $\mu^T(i)$) is within her horizon. □

Tighter Bound on k

We now look whether one could find a tighter bound on k such that for all $k' \geq k$, the singleton set $\{\mu^T\}$ is a horizon-k' vNM stable set. Consider the proof of Theorem 1. We modify Step 1.1.A - 1.1.C as follows.

Step 1.1.A. Let $c_1^1 = (s^1, i^1, ..., s^l, i^l)$. If $\{(i,s)|i,s \in c_1^1 \text{ and } s \mapsto i\} \subseteq \mu$, then go to Step 1.1.B with $\mu'_{1,1} = \mu$. If $\{(i,s)|i,s \in c_1^1 \text{ and } s \mapsto i\} \nsubseteq \mu$, then choose any agent $i \in \{i^1, ..., i^l\}$ such that $s \neq \mu(i) \neq \mu^T(i)$ and $s \mapsto i$ with $i, s \in c_1^1$, say agent i^t. This agent i^t matches with object s^t that ranks her first on its priority list. Next, following the order $i^{t-1}, ..., i^1, i^l, i^{l-1}, ..., i^{t+1}$, we match each agent to the object that ranks her first on its priority list (if not already matched to it). We reach the matching $\mu'_{1,1} = \mu + \{(i,s)|i,s \in c_1^1 \text{ and } s \mapsto i\} - \{(i,\mu(i))|i,s \in c_1^1 \text{ and } s \mapsto i\} - \{(j,s)|j \notin c_1^1, i,s \in c_1^1, s \mapsto i \text{ and } \mu(s) = j\} + \{(j,j)|j \notin c_1^1, i,s \in c_1^1, s \mapsto i \text{ and } \mu(s) = j\}$, where each agent involved in c_1^1 is matched to the object that ranks her first on its priority list, i.e., i^k is matched to s^k for all $k = 1, ..., l$. Step 1.1.A counts at most $\bar{c}_1^1 = \#\{i \in I | i \in c_1^1\}$ steps.

Step 1.1.B. From the matching $\mu'_{1,1}$, let agent i^{t+1} leave her object s^{t+1} so that object s^{t+1} is not matched to any agent. Next, we match s^{t+1} to agent i^t. Now, agent i^t is matched to the object she gets under the TTC mechanism. As agent i^t matches with s^{t+1}, the object s^t is not assigned to any agent. Next, we match s^t to agent i^{t-1}. Now, agent i^{t-1} is matched to the object she gets under the TTC mechanism and the object s^{t-1} is not assigned to any agent. Next, we match s^{t-1} to agent i^{t-2}. Now, agent i^{t-2} is matched to the object she gets under the TTC mechanism and the object s^{t-2} is not assigned to any agent, and so forth until we match s^t to agent i^{t+1}. We have reached $\mu'''_{1,1}$ with $m_1^1 \subseteq \mu'''_{1,1}$ and so agents belonging to c_1^1 are assigned to the same object as in μ^T. If $1 \neq L_1$, then go to Step 1.2. Otherwise, go to Step 1. End with $\mu'''_{1,L_1} = \mu'''_{1,1}$. In Step 1.1.B, each agent obtains her object assigned by the TTC algorithm after at most $\bar{c}_1^1 + 2$ steps from having matched in Step 1.1.A to the object that ranks her first on its priority list.

Step 1.1 originally counts at most $3\bar{c}_1^1 - 1$ steps. As we use now the order in which the agents are matched to the objects that rank them first on their priority lists in Step 1.1.A, there is at most $\bar{c}_1^1 + 2$ steps between the first move of an agent in Step 1.1.A and her final move in Step 1.1.B. Hence, it becomes sufficient that the agents who move in Step 1.1 look forward $\bar{c}_1^1 + 2$ steps ahead to have incentives for engaging the move toward the matching $\mu'''_{1,1}$, where they already get the object assigned by the TTC. Once they reach those matches, they do not move afterwards.

Thus, the lower bound $\underline{k} = \gamma + 2$ is a tighter bound on k such that for all $k' \geq \underline{k}$, the singleton set $\{\mu^T\}$ is a horizon-k' vNM stable set.[30]

Horizon-L Farsighted Set

The notion of a horizon-L farsighted set of matchings is obtained by adapting Herings et al. (2019)'s definition of a horizon-L farsighted set of networks to priority-based matching problems.

Definition 6. Let $\langle I, S, P, F \rangle$ be a priority-based matching problem. A farsighted improving path of length L from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \backslash \{\mu\}$ is a finite sequence of distinct matchings $\mu_0, ..., \mu_L$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, ..., L-1\}$ either

- **i.** $\mu_{l+1} = \mu_l - (i,s) + (i,i)$ for some $(i,s) \in I \times S$ such that $\mu_L(i) P_i \mu_l(i)$, or
- **ii.** $\mu_{l+1} = \mu_l + (i,s) - (i,\mu_l(i)) - \{(j,s)|\mu_l(j) = s\} + \{(j,j)|\mu_l(j) = s\}$ for some $(i,s) \in I \times S$ such that $\mu_L(i) P_i \mu_l(i)$ and $F_s(i) < F_s(j)$ if $\mu_l(s) = j$.

If there exists a farsighted improving path of length L from μ to μ', then we write $\mu \longrightarrow_L \mu'$. For a given matching μ and some $L' \geq 1$, let $\hat{\phi}_{L'}(\mu)$ be the set of matchings that can be reached from μ by a farsighted improving path of length $L \leq L'$. That is, $\hat{\phi}_{L'}(\mu) = \{\mu' \in \mathcal{M} | \exists\, L \leq L' \text{ such that } \mu \longrightarrow_L \mu'\}$. Let $\hat{\phi}_\infty(\mu) = \{\mu' \in \mathcal{M} | \exists\, L \in \mathbb{N} \text{ such that } \mu \longrightarrow_L \mu'\} = \phi_\infty(\mu)$ denote the set of matchings that can be reached from μ by some farsighted improving path. From Lemma 1 in Herings et al. (2019), we have that for every $L \geq 1$, for every $\mu \in \mathcal{M}$, it holds that $\hat{\phi}_L(\mu) \subseteq \hat{\phi}_{L+1}(\mu)$, and that for $L \geq k^\star$, for every $\mu \in \mathcal{M}$, it holds that $\hat{\phi}_L(\mu) = \hat{\phi}_{L+1}(\mu) = \hat{\phi}_\infty(\mu) = \phi_\infty(\mu)$.

The set $\hat{\phi}_L^2(\mu) = \hat{\phi}_L(\hat{\phi}_L(\mu)) = \{\mu'' \in \mathcal{M} | \exists\, \mu' \in \hat{\phi}_L(\mu) \text{ such that } \mu'' \in \hat{\phi}_L(\mu')\}$ consists of those matchings that can be reached by a composition of two farsighted improving paths of length at most L from μ. For $m \in \mathbb{N}$, let $\hat{\phi}_L^m(\mu)$ be the matchings that can be reached from μ by means of m

compositions of farsighted improving paths of length at most L. Let $\widehat{\phi}_L^{\infty}$ denote the set of matchings that can be reached from μ by means of any number of compositions of farsighted improving paths of length at most L.[31]

The notion of a horizon-L farsighted set is based on two main requirements: horizon-L deterrence of external deviations and horizon-L external stability.

Definition 7. For $L \geq 1$, a set of matchings $V \subseteq \mathcal{M}$ satisfies horizon-L deterrence of external deviations if for every $\mu \in V$,[32]

i. $\forall (i,s) \notin \mu$ such that $\widetilde{\mu} = \mu + (i,s) - (i,\mu(i)) - \{(j,s) | \mu(j) = s\} + \{(j,j) | \mu(j) = s\}$ and $\widetilde{\mu} \notin V$, either there exists $\mu' \in [\widehat{\phi}_{L-2}(\widetilde{\mu}) \cap V] \cup [\widehat{\phi}_{L-1}(\widetilde{\mu}) \backslash \widehat{\phi}_{L-2}(\widetilde{\mu})]$ such that $\mu R_i \mu'$, or $F_s(i) > F_s(j)$ if $\mu(s) = j$,

ii. $\forall (i,s) \in \mu$ such that $\widetilde{\mu} = \mu - (i,s) + (i,i)$ and $\widetilde{\mu} \notin V$, there exists $\mu' \in [\widehat{\phi}_{L-2}(\widetilde{\mu}) \cap V] \cup [\widehat{\phi}_{L-1}(\widetilde{\mu}) \backslash \widehat{\phi}_{L-2}(\widetilde{\mu})]$ such that $\mu R_i \mu'$.

Definition 7 captures that forming the match (i,s) at $\mu \in V$ and reaching a matching $\widetilde{\mu}$ outside of V, is deterred by the threat of ending in μ'. Here, μ' is such that either there is a farsighted improving path of length smaller than or equal to $L-2$ from $\widetilde{\mu}$ to μ' and μ' belongs to V or there is a farsighted improving path of length equal to $L-1$ from $\widetilde{\mu}$ to μ' and there is no farsighted improving path from $\widetilde{\mu}$ to μ' of smaller length.

Definition 8. For $L \geq 1$, a set of matchings $V \subseteq \mathcal{M}$ satisfies horizon-L external stability if for every $\mu' \in \mathcal{M} \backslash V, \widehat{\phi}_L^{\infty}(\mu') \cap V \neq \emptyset$.

Definition 9. For $L \geq 1$, a set of matchings $V \subseteq \mathcal{M}$ is a horizon-L farsighted set if it is a minimal set satisfying horizon-L deterrence of external deviations and horizon-L external stability.

Proof of Theorem 2. Take $L \geq 3\gamma$. We show that $\{\mu^T\}$ is a horizon-L farsighted set. First, $\{\mu^T\}$ is a minimal set. Second, $\{\mu^T\}$ satisfies horizon-L deterrence of external deviations. Any deviation from μ^T to $\widetilde{\mu} = \mu^T - (i,\mu^T(i)) + (i,i)$ is deterred as agent i is worse off at $\widetilde{\mu}$, where $\widetilde{\mu}(i) = i$. An agent i may only have incentives to deviate from μ^T to $\widetilde{\mu} = \mu^T + (i,s) - (i,\mu(i)) - \{(j,s) | \mu(j) = s\} + \{(j,j) | \mu(j) = s\}$ if she matches to an object s that was assigned in μ^T to some agent j who belongs to a cycle formed before agent i's cycle in the TTC algorithm. Hence, any deviation from μ^T to $\widetilde{\mu} = \mu^T + (i,s) - (i,\mu(i)) - \{(j,s) | \mu(j) = s\} + \{(j,j) | \mu(j) = s\}$ can be deterred as $\mu^T - (i,\mu^T(i)) + (i,i) \in \widehat{\phi}_{L-1}(\widetilde{\mu})$ and $\mu^T \in \widehat{\phi}_{L-1}(\widetilde{\mu})$. Indeed, from $\widetilde{\mu}$ the agents belonging to agent j's cycle in the TTC algorithm can simply follow the steps of Theorem 1's proof to reach $\mu^T - (i,\mu^T(i)) + (i,i)$, and this farsighted improving path counts at most $3\gamma - 1$ moves. For $L > 3\gamma, \mu^T \in \widehat{\phi}_{L-1}(\widetilde{\mu})$, as agent i has incentives to match to $s = \mu^T(i)$ at $\mu^T - (i,\mu^T(i)) + (i,i)$. Third, the horizon-$k$ improving path from Theorem 1's proof can be decomposed in a succession of farsighted improving paths of length smaller than or equal to $3\gamma - 1$, where each farsighted improving path consists of the formation of the matches between the agents belonging to the same cycle in the TTC algorithm. Hence, for every $\mu \in \mathcal{M} \backslash \{\mu^T\}, \mu^T \in \widehat{\phi}_L^{\infty}(\mu)$, and so $\{\mu^T\}$ satisfies horizon-L external stability. □

Proof of Proposition 1

Proof. As $\{\mu^\star\}$ is a singleton set, internal stability (IS) is satisfied. (ES) Take any matching $\mu \neq \mu^\star$; we need to show that $\bar{\phi}_k(\mu) \ni \mu^\star$ for $k \geq 2n+2$. We build in steps a horizon-k improving path from μ to $\mu^\star$ for $k \geq 2n+2$. We consider two cases.

1. Take $\mu \neq \mu^\star$ such that the couple $(i^\star, c(i^\star))$ does not obtain their most preferred matches. From μ, looking forward k steps ahead, agent $i^\star$ matches successively with each object $s \neq \mu^\star(i^\star)$ that was assigned to some agent $j \neq c(i^\star)$ under μ (i.e., $\mu(s) \in I, \mu(s) \neq c(i^\star)$) to reach μ'. Next, agent $i^\star$ matches with $\mu^\star(i^\star)$ to reach the matching μ'', where $\mu''(i^\star) = \mu^\star(i^\star), \mu''(c(i^\star)) = \mu(c(i^\star))$ and $\mu''(i) = i$ for all $i \in I, i \neq i^\star, c(i^\star)$. Next, if $\mu''(c(i^\star)) \neq \mu^\star(c(i^\star))$, agent $c(i^\star)$ matches with $\mu^\star(c(i^\star))$. Next, each $i \in I$ $(i \neq i^\star)$ matches one by one to the object she is assigned to in $\mu^\star$. Hence, $\mu^\star \in \bar{\phi}_k(\mu)$.
2. Take $\mu \neq \mu^\star$ such that the couple $(i^\star, c(i^\star))$ obtains their most preferred matches. As $\mu^\star$ is Pareto-efficient, there is some couple $(j, c(j))$ that prefers $\mu^\star$ to μ. (2.a) If $\mu(j) \neq j$ (or $\mu(c(j)) \neq c(j)$) then, looking forward k step ahead, agent $i^\star$ and agent j if $\mu(j) \neq j$ (or agent $c(j)$ if $\mu(c(j)) \neq c(j)$) give up their objects to reach μ', where $\mu'(i^\star) = i^\star$ and $\mu'(j) = j$. The couple $(j, c(j))$ strictly prefers $\mu^\star$ to μ, whereas couple $(i^\star, c(i^\star))$ is indifferent. In $\mu' \neq \mu^\star$, the couple $(i^\star, c(i^\star))$ does not obtain their most preferred matches, and so we proceed as in (1) to reach $\mu^\star$. Hence, $\mu^\star \in \bar{\phi}_k(\mu)$. (2.b) Assume now that there is no couple that prefers $\mu^\star$ to μ and one of the partner is assigned to some object in μ. Therefore, $\mu(j) = j$ and $\mu(c(j)) = c(j)$ for every couple $(j, c(j))$ that prefers $\mu^\star$ to μ (i.e., both members of couple $(j, c(j))$ are unassigned in μ). Then, there is at least one agent j and some unassigned object s in μ (i.e., $\#\{i \in I | (i,s) \in \mu\} = 0$) such that the couple $(j, c(j))$ prefers $\mu^\star$ to $\mu' = \mu + (j,s)$. Otherwise, $\mu^\star$ would be Pareto-dominated, contradicting the assumption that $\mu^\star$ is Pareto-efficient. In $\mu' \neq \mu^\star$, the couple $(i^\star, c(i^\star))$ obtains their most preferred matches and $\mu'(j) \neq j$ for some couple $(j, c(j))$ that prefers $\mu^\star$ to μ'. From μ' we proceed as in (2.a) to reach $\mu^\star$. Hence, $\mu^\star \in \bar{\phi}_k(\mu)$. □

Proof of Theorem 3

Proof. From the proof of Theorem 1, it follows that $\{\mu^T\}$ is a horizon-k vNM stable set for $k \geq 3\gamma - 1$.[33] We now show that $\{\mu^T\}$ is the unique horizon-k vNM stable set for $k \geq 3\gamma - 1$ as $\bar{\phi}_k(\mu^T) = \emptyset$. Consider any cycle that is obtained in the first step of the TTC algorithm. All the agents involved in this cycle obtain their most preferred object in μ^T and, in this cycle, any agent obtains the endowment of another agent who is also in the cycle. Hence, from μ^T, they will never engage a move toward another matching. Consider now any cycle that is obtained in the second step of the TTC algorithm. Taking as fixed the matches done in μ^T by all agents involved in any cycle of the first step of the TTC, all the agents involved in this cycle of the second step of the TTC obtain their most preferred object in μ^T and, in this cycle, any agent obtains the endowment of another agent who is also in the cycle. Knowing that agents from any cycle of the first step of the TTC will never engage a move, agents from any cycle of the second step of the TTC will never engage either a move from μ^T toward another matching. Repeating this argument with the matches found in steps 3, 4, ... of the TTC leads to the conclusion that $\bar{\phi}_k(\mu^T) = \emptyset$. Hence, any set $V \neq \{\mu^T\}$ would violate (ES) or (IS), and so $\{\mu^T\}$ is the unique horizon-k vNM stable set for $k \geq 3\gamma - 1$. □